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# Absolute and convective instabilities of spatially periodic flows

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## Contents



Stability of monochromatic waves and wave packet evolution are of fundamental importance in the transition to turbulence in open-flow systems. Although the analysis of monochromatic wave stability and the evolution of linear wave packets in spatially homogeneous or parallel flows is generally well understood, such analysis for spatially inhomogeneous flows is not so well understood. In this paper we consider the problem of linear wave packet evolution in a spatially periodic medium. The mathematical formalism of absolute and convective instabilities in spatially homogeneous flows is generalized to the spatially periodic case. The Laplace transform is used to reduce the initial-value problem to a system of ordinary differential equations with periodic coefficients which is then completely analysed using the Floquet theory and the Fourier transform. We define a generalized dispersion relation by $\Delta(\mu,\omega)=0$, where $\mu \in \mathbb{C}\backslash\{0\}$ is a spatial Floquet multiplier and $\omega \in \mathbb{C}$ is a frequency (and a Laplace transform parameter). We find that a spatially periodic flow is absolutely unstable if and only if $\Delta(\mu,\omega)$ has a double root (or more generally a multiple root) in $\mu$ at $(\mu_0,\omega_0)$ with $\operatorname{Im}\omega_0 > 0$ that satisfies the collision criterion: i.e. the double root splits under perturbation in such a way that in the limit, as $\operatorname{Im}(\omega-\omega_0) \to +\infty$, with $\operatorname{Re}(\omega-\omega_0)=0$, one of the roots goes interior and the other exterior to the unit circle $\{|\mu|=1\}$ in the complex $\mu$-plane. Further results are obtained on the long-time asymptotics of absolutely unstable and absolutely stable flows and the signalling problem for periodic media is introduced. In general the asymptotic states of unstable periodic waves are quasi-periodic in space and/or time. The theory is applied to the finite-amplitude periodic travelling wave states of the real and complex Ginzburg–Landau equation. We find that the Eckhaus instability is always absolutely unstable whereas in the complex case there is an interesting decomposition of the region of unstable finite-amplitude travelling waves into an absolutely unstable region and a convectively unstable but absolutely stable region. The problem of unstable wave packets for the Navier–Stokes equations, linearized about a spatially periodic state, is also formulated.

## 1. Introduction

In fluid flows, the linear stability theory for a monochromatic wave in an open system is well developed, leading, for parallel flows, to the Orr–Sommerfeld equation (cf. Drazin & Reid 1981). However, in order to understand the long-time asymptotic response of a homogeneous flow a wave-packet analysis and a consideration of the initial-value problem is necessary. In fluid mechanics, the evolution of linear wave packets in spatially homogeneous flows in open systems has been an area of active interest since the early 1960s because of the significance of such waves in the transition to turbulence (cf. Benjamin 1961; Gaster 1968, 1975; Gaster & Grant 1975; Drazin & Reid 1981, § 47; Infeld & Rowlands 1990, §§ 2–4).

In this paper a theory for the long-time asymptotics of linear wave packets in a spatially periodic medium or flow is developed. Such a medium or flow arises when the governing system of equations has a spatially periodic state or spatially periodic travelling wave about which the equations are linearized. To set the ideas we consider the following example for expository purposes:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \qquad x \in \mathbb{R}, \quad t > 0, \tag{1.1}$$

where $u$ is a real-valued function of $x$ and $t$ and $f : \mathbb{R} \to \mathbb{R}$ is a smooth function. We suppose $f(u)$ is such that $u_{xx} + f(u) = 0$ has a periodic solution of period $L > 0$. Let us denote this solution by $\overline{u}(x)$ and linearize (1.1) about this state

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f'(\overline{u})u. \tag{1.2}$$

Since the coefficients in (1.2) are independent of time, we can consider the ansatz $u(x,t) = \mathrm{e}^{\lambda t}\widehat{u}(x)$, with

$$\frac{\mathrm{d}^2\widehat{u}}{\mathrm{d}x^2} + f'(\overline{u})\widehat{u} = \lambda\widehat{u}, \tag{1.3}$$

and, classically, one would state the linear stability problem for the basic state $\overline{u}(x)$ as follows. If there exists a solution $(\widehat{u}(x), \lambda)$ of (1.3), with $\widehat{u}(x)$ bounded for all $x \in \mathbb{R}$ and $\operatorname{Re}\lambda > 0$, then the basic state is linearly unstable. Conversely, if for every solution $(\widehat{u}(x), \lambda)$ of (1.3), with $\widehat{u}(x)$ bounded for all $x \in \mathbb{R}$, the exponent satisfies $\operatorname{Re}\lambda \leqslant 0$ we say that the basic state in linearly stable. When $\overline{u}(x)$ is $L$-periodic the analysis of the problem is considerably simplified, since, in that case, every bounded solution of (1.3) can be expressed, via Floquet's theorem, as a product of an $L$-periodic function and an exponential with purely imaginary exponent. In fact, for the example (1.3), it is easy to show, using the theory of Hill's equation or Floquet theory, that any spatially periodic basic state of (1.1) is linearly unstable. However, the stable–unstable classification is insufficient in an open system as unstable disturbances that decay suitably at infinity in space can behave in dramatically different ways as $t \to \infty$. To clarify this point we first review the theory for the spatially homogeneous case and then indicate how it is generalized in this paper to the case when the basic state is spatially periodic.

For this purpose we consider the linear partial differential equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + u, \qquad x \in \mathbb{R}, \quad t > 0, \tag{1.4}$$

where $a \in \mathbb{R}$ and $u$ is real valued. It is easy to see by making a normal-mode ansatz,

$$u(x,t) = A_0 \mathrm{e}^{\mathrm{i}(kx-\omega t)}, \tag{1.5}$$

that the basic state, the trivial state in this example, is linearly unstable, since the frequency $\omega$ satisfies

$$\omega = -ak + \mathrm{i}(1-k^2) \tag{1.6}$$

and, therefore, $\operatorname{Im}\omega = \omega_{\mathrm{i}}(k) = 1-k^2$ for real $k$. There is a band of unstable wavenumbers in the interval $-1 < k < 1$. From a normal mode perspective the linear stability analysis is complete: the basic state is linearly unstable. However, the long-time asymptotics of an initial pulse will differ, dependent on the value of $a$, rather dramatically. These asymptotics in the example (1.4) can be clearly seen using the exact solution.

At $t = 0$ we take the initial data for (1.4) to be a Gaussian pulse of unit amplitude

$$u(x,t)\big|_{t=0} = \mathrm{e}^{-bx^2/4}, \qquad b > 0. \tag{1.7}$$

Then the exact solution of (1.4) for all $x \in \mathbb{R}$ and all $t \geqslant 0$ is

$$u(x,t) = \frac{\mathrm{e}^t}{\sqrt{1+bt}} \exp\left[-\frac{b(x+at)^2}{4(1+bt)}\right]. \tag{1.8}$$

Although the trivial state is unstable, it can be seen from (1.8) that the growth is not uniform in $x$. For instance, the exact solution at $x = 0$ is

$$\begin{aligned} u(x,t)\big|_{x=0} &= \frac{\mathrm{e}^t}{\sqrt{1+bt}} \exp\left[-\frac{ba^2t^2}{4(1+bt)}\right] \\ &= \frac{1}{\sqrt{1+bt}} \exp\left\{\frac{t}{1+bt}[1+bt(1-\tfrac{1}{4}a^2)]\right\}. \end{aligned} \tag{1.9}$$

As $t \to \infty$ the solution at $x = 0$ decays exponentially to zero if $|a| > 2$. This is also true for any *fixed* value of $x$. However, in a frame of reference moving at speed $-a$ the solution (1.8) is exponentially growing in time for any $a \in \mathbb{R}$. In other words, if the convection velocity $|a|$ is strong enough the instability in the absolute frame of reference is localized in space. Such an instability is called a *convective instability* or *localized instability*.

On the other hand if $|a| < 2$ there is an exponential growth of the solution (1.8) at $x = 0$ (cf. (1.9)) and at every fixed $x$ position in the absolute frame of reference. Such an instability is, therefore, called an *absolute instability* or *global instability*. The





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distinction between these two types of instabilities is of fundamental importance in applications. A convective or localized instability can convect away from a region of interest and it may be possible to control. An absolute or global instability is generally catastrophic, growing exponentially in time at all points in space leading to immediate and dramatic changes in the flowfield.

The above distinction, in the example (1.4), was made using the exact solution which is not in general available and, therefore, an easily applicable criterion is necessary to distinguish between the two types of instability. The importance of such a criterion in the spatially homogeneous case and its precise formulation seem to have been first recognized in the plasma physics literature (cf. Twiss 1951; Landau & Lifshitz 1953; Briggs 1964; and references therein). There has been considerable generalization and application of the theory since (cf. Bers 1983; Huerre & Monkewitz 1985; 1990; Koch 1985; Brevdo 1988, 1991, 1992; Infeld & Rowlands 1990, ch. 3; Brevdo *et al.* 1994; and references therein).

In the most common case, a necessary, *but not sufficient*, condition for absolute instability of a spatially homogeneous flow is the existence of a saddle point of the frequency function $\omega = \omega(k)$ in the complex $k$-plane for some $\omega_0 \in \mathbb{C}$ with $\operatorname{Im}\omega_0 > 0$, where the normal mode disturbance is of the form $e^{i(kx-\omega t)}$. However, for sufficiency it is required that the two $k$-roots of the dispersion relation function $D(k,\omega)$ that collide to form the saddle point move to opposite sides of the real $k$-axis in the limit as $\operatorname{Im}(\omega-\omega_0) \to +\infty$, for $\operatorname{Re}(\omega-\omega_0) = 0$. This is the pinching condition (Briggs 1964). To give meaning to this condition it is necessary to treat the initial-value problem using the Fourier–Laplace (FL) transform. Analysis of a monochromatic wave using the normal-mode approach, alone, is insufficient for a classification of the long-time asymptotics. For example, application of the FL transform (see (2.7) and (3.13) for the precise definition of the Laplace and Fourier transformations used here) to (1.4) results in

$$D(k,\omega)u(k,\omega) = u_0(k), \quad \text{where} \quad D(k,\omega) = -i\omega + k^2 - iak - 1, \tag{1.10}$$

and where $u(k,\omega)$ is the FL transformation of $u(x,t)$ and $u_0(k)$ is the Fourier transformation of the initial condition. The general solution of the initial value problem formulated for (1.4) can then be written abstractly in terms of the inverse FL transformation as

$$u(x,t) = \frac{1}{4\pi^2}\int_{i\sigma-\infty}^{i\sigma+\infty}\int_{-\infty}^{\infty} \frac{u_0(k)}{D(k,\omega)} e^{i(kx-\omega t)}\,dk\,d\omega. \tag{1.11}$$

The Bromwich contour $\{\omega \in \mathbb{C} \mid \operatorname{Im}(\omega) = \sigma \text{ and } \operatorname{Re}\omega \in \mathbb{R}\}$ for the inverse Laplace transformation is chosen so that $\sigma$ is greater than the largest value of $\operatorname{Im}\omega$ for all $k \in \mathbb{R}$ with $D(k,\omega) = 0$. Throughout the paper $\sigma$ is assumed to be finite.

To complete the analysis of the example (1.4) we verify the necessary and sufficient conditions for absolute instability. The saddle point condition requires $D(k,\omega) = D_k(k,\omega) = 0$. From (1.10) we have that $D_k = 2k - ia$. Therefore, there is a saddle point at

$$k_0 = \tfrac{1}{2}ia \quad \text{and} \quad \omega_0 = i(1 - \tfrac{1}{4}a^2). \tag{1.12}$$

The saddle point $k_0$ corresponds to a frequency $\omega_0$ with $\operatorname{Im}\omega_0 > 0$ when $|a| < 2$. For sufficiency we must check the pinching condition. However, one easily verifies that $D(k,\omega) = 0$ implies that

$$(k-k_0)^2 = i(\omega-\omega_0) \quad \text{and} \quad k_\pm = k_0 \pm \sqrt{i(\omega-\omega_0)}. \tag{1.13}$$

Therefore, $\mathrm{Im}(k_\pm - k_0) \to \pm\infty$ as $\mathrm{Im}(\omega - \omega_0) \to +\infty$, with $\mathrm{Re}(\omega - \omega_0) = 0$, showing that there is an absolute instability for $|a| < 2$.

The difficulty with extending the theory of wave-packet evolution to the case when the basic state is spatially inhomogeneous is that the Fourier transform does not result in a separation of the spatial variable when the coefficients are non-constant. An absolute-convective or global-local classification of instabilities when the basic state varies weakly with $x$, using the WKB method or a quasi-homogeneous hypothesis, has been considered by several authors (cf. Koch 1985; the review articles by Bers (1983) and Huerre & Monkewitz (1990) and the analysis of Hunt & Crighton (1991)). Polovin & Demutskii (1981) have given a sufficient condition for absolute instability of a periodic basic state for a particular class of problems by using a Green's function approach. However, a general theory for classifying absolute and convective instabilties in spatially inhomogeneous media has never been constructed. In this paper a general theory for wave packets in inhomogeneous media when the inhomogeneity is spatially periodic is presented. In addition to precise necessary and sufficient conditions for absolutely unstable wave packets, we present precise conditions for convective instability in spatially periodic media, the asymptotic behaviour of the wave packet along all possible rays $x = x_o + Vt$ and, moreover, treat the signalling problem in periodic media with a moving source. A complete theory is possible for spatially periodic media because of the Floquet decomposition. The theory relies also on the Laplace transform in time and a Fourier transform in space coupled with the Floquet decomposition.

We rewrite the example (1.1) in the form

$$\boldsymbol{M}Z_t + Z_x = F(Z), \quad \text{where} \quad Z = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \text{with} \quad v = u_x. \tag{1.14}$$

Here

$$\boldsymbol{M} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad F(Z) = \begin{pmatrix} v \\ -f(u) \end{pmatrix}. \tag{1.15}$$

Then a spatially periodic state of (1.1) is represented in the formulation (1.14) as an $L$-periodic solution of the system of ordinary differential equations $W_x = F(W)$ with $W(x + L) = W(x)$. In general, we will develop a theory in the sequel for classifying linear instabilities of spatially periodic states of PDEs of the form (1.14), when $Z \in \mathbb{R}^N$ with $N$ arbitrary but finite. In fact, the general theory proceeds using the abstract form (1.14), with suitable generalization of the operators $\boldsymbol{M}$ and $F(Z)$, but it is useful to keep in mind the example (1.1), and the examples in §§ 6–8. In §§ 6–8 the Ginzburg–Landau equation, Navier–Stokes equations, Fitzhugh–Nagumo equations and other physical models are formulated as in (1.14).

Linearizing (1.14) about the $L$-periodic state $W(x)$ gives

$$\boldsymbol{M}Z_t + Z_x = \boldsymbol{A}(x)Z, \quad \text{with} \quad \boldsymbol{A}(x) = DF(W(x)). \tag{1.16}$$

A formal application of the Laplace transform to (1.16) results in

$$Z_x(x, \omega) = [\boldsymbol{A}(x) + \mathrm{i}\omega\boldsymbol{M}]Z(x, \omega) + g(x, \omega), \tag{1.17}$$

where $g(x, \omega) = \boldsymbol{M}Z(x, t)|_{t=0}$ plus any source term. The complete analysis of (1.17) using the Floquet theory and the Fourier transform is given in § 3. The generalization to the spatially periodic case of the necessary and sufficient conditions for absolute and convective instability is summarized in terms of the Floquet multipliers as follows.

Let $\boldsymbol{\Phi}(x,\omega)$ be the flow operator for the homogeneous part of the ODE in (1.17), that is, $\boldsymbol{\Phi}_x = [\boldsymbol{A}(x) + \mathrm{i}\omega\boldsymbol{M}]\boldsymbol{\Phi}$, that satisfies $\boldsymbol{\Phi}(0,\omega) = \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix. Then the monodromy operator for the system is defined by $\boldsymbol{\Phi}(L,\omega)$. The Floquet multipliers are

$$\Sigma(\boldsymbol{\Phi}(L,\omega)) = \{\,\mu \in \mathbb{C} \,|\, \Delta(\mu,\omega) = \det\,[\boldsymbol{\Phi}(L,\omega) - \mu\boldsymbol{I}] = 0\}. \tag{1.18}$$

The flow is absolutely unstable if and only if the function $\Delta(\mu,\omega)$ has a multiple root in $\mu$ for $(\mu_0,\omega_0)$, with $\operatorname{Im}\omega_0 > 0$, which satisfies the following pinching criterion. Under perturbation of the imaginary part of the frequency $\omega_0$ the multiple Floquet multiplier $\mu_0$ in (1.18) should split in such a way that at least one of the Floquet multipliers should lie interior to the unit circle and at least one multiplier exterior to the unit circle in the complex $\mu$-plane in the limit as $\operatorname{Im}(\omega - \omega_0) \to +\infty$, with $\operatorname{Re}(\omega - \omega_0) = 0$.

The plan of the paper is as follows. In § 2 the linear stability problem is formulated and in § 3 the Floquet analysis and Fourier transform of systems of the form (1.17) are presented. In § 4 the asymptotic evaluation of the solutions for $t \to \infty$ and the formal classification of instabilities are made. We find that the oscillatory part of the dominant contribution to the instability as $t \to \infty$, when the observer is in a moving frame, is in general quasi-periodic.

In the case when the flow is absolutely stable but convectively unstable an interesting question is the response due to an externally imposed time-periodic source. In § 5 a theory for the response of a spatially periodic flow to a time-periodic forcing is presented. The wavy part of the dominant term in the asymptotic response will in general be quasi-periodic in space, even when there is only one unstable contribution. There are in general two wavelengths in a perturbation class; the wavelength of the basic state and the second wavelength associated with the Floquet exponent of the perturbation. A response of a travelling wave to a periodic forcing is also generally quasi-periodic in time.

In § 6 the theory is applied to classify the instabilities of spatially periodic states of the complex Ginzburg–Landau (CGL) equation. In addition to being an important equation in applications the advantage of the CGL equation is that the Floquet decomposition can be explicitly constructed leading to the exact conditions for the movement of points (saddle points in this case) contributing to the instabilities and to the classification of the instabilities of finite-amplitude travelling wave solutions. Our main observations are that the Eckhaus instability, modelled by the GL equation with real coefficients, is an absolute instability, whereas in the CGL equation the instabilities of travelling wave states can be either convective or absolute depending on the parameters in the equation and the position along a branch. We present results for a particular example where, along a branch of finite-amplitude travelling waves, there is a transition from absolute to convective instability and then to stability. The consequence of the result in § 6 for physical systems, such as near-critical hydrodynamical stability problems modelled by a CGL equation, is shown in figure 1.

In figure 1 $R$ is a control parameter such as a Reynolds number and $\alpha$ is a wavenumber. The outside parabola is the neutral curve for a basic spatially homogeneous or parallel flow. The shaded region is a projection onto the $(R,\alpha)$-plane of the region of stable finite-amplitude periodic travelling waves obtained by normal-mode analysis. The result of § 6 shows that the domain of unstable finite-amplitude periodic travelling waves is composed of two distinct regions, where the waves are respectively absolutely unstable and absolutely stable but convectively unstable with a

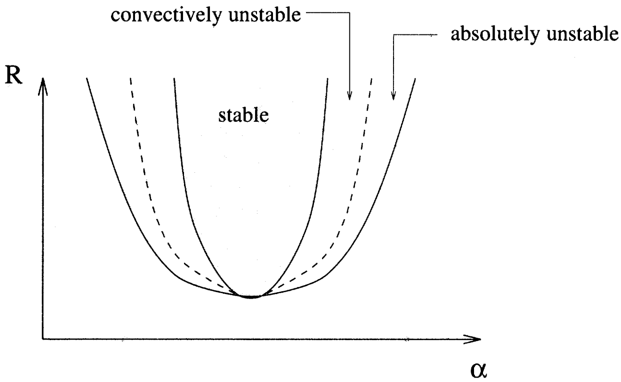


Figure 1. Decomposition of the unstable region, for spatially periodic travelling waves of the complex Ginzburg–Landau equation, into a convectively and absolutely unstable region, in the $(R, \alpha)$ plane where $R$ is a control parameter.

well-defined transition between the two cases. The analysis of the the CGL equation is indicative of how the investigation would proceed with more difficult problems.

A challenging problem in hydrodynamic stability theory is the secondary instability problem, where a finite-amplitude branch of spatially periodic wave solutions, travelling in a shear flow that satisfies the Navier–Stokes equations, loses stability. A classification of the long-time asymptotics of this instability would be of great interest in understanding the transition to turbulence in shear flows. Although the normal mode instability of this problem is generally understood, a wavepacket approach has not been previously considered. In § 7 we outline how the theory of this paper could contribute towards an analysis of this problem. Finally, in § 8 some concluding remarks are made.

## 2. Formulation of the linear stability problem

Without giving here a description of a particular physical problem we assume that the nonlinear dynamics of an unsteady flow can be described by a partial differential equation on the real line of the form

$$\boldsymbol{M}\widehat{Z}_t + \widehat{Z}_x = F(\widehat{Z}), \quad \text{with} \quad x \in \mathbb{R} \quad \text{and} \quad t > 0, \tag{2.1}$$

where $\widehat{Z}$ is a vector-valued function of $(x, t)$, $\boldsymbol{M}$ is a linear operator with constant coefficients and $F(\cdot)$ is a nonlinear operator. In the system (2.1) the equations contain only first order derivatives in space and time and there are no terms with derivatives with respect to $x$ and $t$ in $F(\widehat{Z})$. The form (2.1) is convenient for the analysis, particularly the Floquet theory, and many PDEs appearing in applications can be written in the form (2.1) by introducing additional variables (cf. the applications in §§ 6–8). In the present paper it will be assumed that $\widehat{Z} \in \mathbb{R}^N$ in which case $\boldsymbol{M}$ is an $N \times N$ matrix and $F(\widehat{Z})$ is a nonlinear function on $\mathbb{R}^N$. Some remarks towards the case of a flow with shear, for instance when (2.1) represents the Navier–Stokes equations, are given in § 7.

The linear stability problem for spatially periodic solutions – or spatially periodic media or flow; the terms will be used interchangeably – of (2.1) will be formulated. Let us suppose that there exists an $L$-periodic solution of (2.1), denoted by $W(x)$, with $W(x+L) = W(x)$ and

$$\text{either} \quad W_x = F(W) \quad \text{or} \quad (\boldsymbol{I}_N - c\boldsymbol{M})W_x = F(W), \tag{2.2}$$

where $\boldsymbol{I}_N$ is the $N \times N$ identity matrix and $c$ is a real number. The second case, when $c \neq 0$, corresponds to a state which is periodic relative to a frame moving at speed $c$. The parameter $c$ does not enter the analysis explicitly and, therefore, the explicit dependence of $W(x)$ on $c$ will be suppressed. However, when classifying instabilities, the distinction between different frames of reference is of fundamental importance (cf. the analysis of the GL equation in § 6).

In order to study the linear stability problem we substitute $\widehat{Z} = W + Z$ into (2.1) and linearize about $W$ to obtain

$$\boldsymbol{M}Z_t + Z_x = \boldsymbol{A}(x)Z, \quad \text{with} \quad x \in \mathbb{R} \quad \text{and} \quad t > 0. \tag{2.3}$$

The linear operator $\boldsymbol{A}(x) = DF(W(x))$, where $W(x)$ is the $L$-periodic basic state, and, therefore, $\boldsymbol{A}(x+L) = \boldsymbol{A}(x)$. We will also consider the problem where an external source, represented by a vector-valued function $G(x,t)$, is applied to the wave

$$\boldsymbol{M}Z_t + Z_x = \boldsymbol{A}(x)Z + G(x,t), \quad \text{with} \quad x \in \mathbb{R} \quad \text{and} \quad t > 0. \tag{2.4}$$

The operator $\boldsymbol{A}(x)$ is assumed to depend smoothly on $x$. The function $G(x,t) \in C_0^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{R}^N)$, where $C_0^\infty$ indicates $C^\infty$ functions with compact support. The requirements on $G(x,t)$ can be relaxed to include all source term functions for which the integrals appearing in the formalism converge. The linear problem (2.4) is supplemented by the initial condition

$$Z(x,t)_{|t=0} = Z_0(x), \quad \text{with} \quad Z_0(x) \in C_0^\infty(\mathbb{R}, \mathbb{R}^N), \tag{2.5}$$

and the boundary conditions

$$\lim_{x \to \pm\infty} \|Z(x,t)\| = 0 \quad \text{for each} \quad t > 0. \tag{2.6}$$

The solutions $Z(x,t)$ of (2.4)–(2.6) with at most exponential growth in time are sought in the space $Z(\cdot,t) \in C^\infty(\mathbb{R}, \mathbb{R}^N) \bigcap L^1(\mathbb{R}, \mathbb{R}^N)$. The norm $\|\cdot\|$ represents the Euclidian norm in $\mathbb{R}^N$. Note that the formulation (2.4)–(2.6) could also be associated with a linear problem, that is when $F(Z) = \boldsymbol{A}(x)Z$ in (2.1), where the $x$-dependence of $\boldsymbol{A}(x)$ is due to a material inhomogeneity. For instance, in the case of a flow in a channel with spatially periodic walls.

We are interested in studying the asymptotic behaviour of the perturbations $Z(x,t)$ along the family of rays $x = x_0 + Vt$ for $t \to \infty$, where $x_0$ is constant and $V$ is a ray velocity. If relative to the absolute frame of reference the solution $Z(x_0,t)$ of (2.4)–(2.6) is unbounded for each $x_0$ as $t \to \infty$ the flow is called absolutely unstable. Otherwise an unstable flow is called absolutely stable, but convectively unstable. When the basic state is homogeneous, that is, when $\boldsymbol{A}(x)$ is independent of $x$, and the basic state is unstable, there exists a ray $x = x_0 + Vt$ such that $Z(x_0+Vt,t)$ is unbounded as $t \to \infty$. Therefore, in the homogeneous case, an unstable but absolutely stable state is always convectively unstable. We shall exhibit a similar distinction between asymptotic behaviours when the basic state is spatially periodic.

The Laplace transform, defined by

$$L\{w\}(\omega) = \widetilde{w}(\omega) = \int_0^\infty w(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t, \quad w(t) = \frac{1}{2\pi}\int_{\mathrm{i}\sigma-\infty}^{\mathrm{i}\sigma+\infty} \widetilde{w}(\omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega, \tag{2.7}$$

is applied to (2.4)–(2.6) resulting in

$$Z_x(x,\omega) = [\boldsymbol{A}(x) + \mathrm{i}\omega \boldsymbol{M}]Z(x,\omega) + g(x,\omega), \tag{2.8}$$

where

$$g(x,\omega) = G(x,\omega) + \boldsymbol{M}Z_0(x). \tag{2.9}$$

Here and further in the text we omit the tilde for convenience. The dependent variables are distinguished from their transforms by the independent variables. The boundary conditions for $Z(x,\omega)$ are

$$\lim_{x\to\pm\infty} \|Z(x,\omega)\| = 0 \tag{2.10}$$

for $\omega$ with $\operatorname{Im}\omega$ positive and large enough. In addition, for every such fixed $\omega$ it holds that $Z(x,\omega) \in C^\infty(\mathbb{R},\mathbb{R}^N)\bigcap L^1(\mathbb{R},\mathbb{R}^N)$ in $x$. The meaning of this requirement will become clear in the sequel. The system (2.8) is an inhomogeneous linear system of ordinary differential equations with coefficients depending periodically on $x$. In the next section we shall give a formal solution of (2.8)–(2.10) using the Floquet theory and the Fourier transform in space.

## 3. Floquet theory and the boundary-value problem

Let $\boldsymbol{\Phi}(x,\omega)$ be an $N\times N$ matrix that satisfies the matrix differential equation

$$\boldsymbol{\Phi}_x(x,\omega) = [\boldsymbol{A}(x) + \mathrm{i}\omega \boldsymbol{M}]\boldsymbol{\Phi}(x,\omega) \tag{3.1}$$

and the initial condition

$$\boldsymbol{\Phi}(0,\omega) = \boldsymbol{I}_N. \tag{3.2}$$

The system of equations (3.1) is the homogeneous system associated with the system (2.8). Since the coefficients in (3.1) are continuous $L$-periodic functions of $x$, according to the Floquet theory, the flow operator $\boldsymbol{\Phi}(x,\omega)$ has a decomposition for all $x\in\mathbb{R}$ of the form

$$\boldsymbol{\Phi}(x,\omega) = \boldsymbol{Q}(x,\omega)\mathrm{e}^{\boldsymbol{B}(\omega)x}, \tag{3.3}$$

where $\boldsymbol{Q}(x,\omega)$ is a non-singular, differentiable, $L$-periodic $N\times N$ matrix function of $x$, i.e. $\boldsymbol{Q}(x+L,\omega) = \boldsymbol{Q}(x,\omega)$, that satisfies $\boldsymbol{Q}(0,\omega) = \boldsymbol{I}_N$, and $\boldsymbol{B}(\omega)$ is $N\times N$ matrix (Yakubovich & Starzhinskii 1975). Since the coefficients in (3.1) are $C^\infty$ in $x$ and analytic in $\omega$, it follows that $\boldsymbol{\Phi}(x,\omega)$ is also.

The matrix $\boldsymbol{B}(\omega)$ is defined using the monodromy operator $\boldsymbol{\Phi}(L,\omega)$. Since $\boldsymbol{Q}(x,\omega)$ in (3.3) is $L$-periodic in $x$, it follows that $\boldsymbol{Q}(L,\omega) = \boldsymbol{I}_N$ and, therefore,

$$\boldsymbol{\Phi}(L,\omega) = \mathrm{e}^{\boldsymbol{B}(\omega)L}. \tag{3.4}$$

Now,

$$\det \boldsymbol{\Phi}(x,\omega) = \exp\left(\int_0^x \mathrm{tr}[\boldsymbol{A}(s) + \mathrm{i}\omega\boldsymbol{M}]\,\mathrm{d}s\right), \tag{3.5}$$

where $\mathrm{tr}[\cdot]$ indicates the trace of the matrix in $[\cdot]$. Therefore, $\det\boldsymbol{\Phi}(x,\omega)\neq 0$ for all $x$ and $\omega$. Since $\boldsymbol{\Phi}(0,\omega) = \boldsymbol{I}_N$, for every complex $\omega$ there exists a positive real number $\delta(\omega)$ such that for $|x|<\delta(\omega)$ it holds that $\|\boldsymbol{\Phi}(x,\omega) - \boldsymbol{I}_N\| < \frac{1}{2}$. For all pairs $\{(x,\omega)|\ \omega\in\mathbb{C},\ |x|<\delta(\omega)\}$ we can define a branch of $\ln\boldsymbol{\Phi}(x,\omega)$ which is $C^\infty$ in $x$





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and analytic in $\omega$

$$\ln \boldsymbol{\Phi}(x,\omega) = \ln\{\boldsymbol{I}_N + [\boldsymbol{\Phi}(x,\omega) - \boldsymbol{I}_N]\} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}[\boldsymbol{\Phi}(x,\omega) - \boldsymbol{I}_N]^n. \tag{3.6}$$

For every disc $\{|\omega| \leqslant r_0\}$ in the complex $\omega$-plane there exists a positive number $\delta_1(r_0)$ such that the logarithm (3.6) is analytic in this disc for all $|x| < \delta_1(r_0)$. This is due to the compactness of the disc. The matrix $\ln \boldsymbol{\Phi}(L,\omega)$ exists for all $\omega$ because $\det \boldsymbol{\Phi}(L,\omega) \neq 0$ for all $\omega$, and at every $\omega$ at which the $\boldsymbol{\Phi}(L,\omega)$ has $N$ distinct eigenvalues the separate branches of the matrix function $\ln \boldsymbol{\Phi}(L,\omega)$ are analytic.

Based on the above observations it is reasonable to conjecture that a branch of $\ln \boldsymbol{\Phi}(L,\omega)$ is analytic in the entire complex plane and it would be interesting to give a proof of analyticity of $\ln \boldsymbol{\Phi}(L,\omega)$ in general, but we do not consider this question here. In the examples considered in § 6 it is shown explicitly that $\ln \boldsymbol{\Phi}(L,\omega)$ is an analytic function for all $\omega \in \mathbb{C}$, even though $\boldsymbol{\Phi}(L,\omega)$ has multiple eigenvalues for particular values of $\omega$. Under the above hypothesis, we have that the matrix function

$$\boldsymbol{B}(\omega) = \frac{1}{L} \ln \boldsymbol{\Phi}(L,\omega) \tag{3.7}$$

is analytic for all $\omega$. Hence, $\boldsymbol{Q}(x,\omega) = \boldsymbol{\Phi}(x,\omega)\mathrm{e}^{-\boldsymbol{B}(\omega)x}$ is an entire function of $\omega$.

The spectrum of $\boldsymbol{B}(\omega)$ is denoted by

$$\Sigma[\boldsymbol{B}(\omega)] = \{\mathrm{i}k_j(\omega) \in \mathbb{C},\ 1 \leqslant j \leqslant N \mid \det[\boldsymbol{B}(\omega) - \mathrm{i}k_j(\omega)\boldsymbol{I}_N] = 0, \text{ with } \mathrm{i}^2 = -1\} \tag{3.8}$$

including multiple eigenvalues. We define the subset of unstable wave numbers by

$$\Sigma_u[\boldsymbol{B}(\omega)] = \{\mathrm{i}k_j(\omega) \in \Sigma[\boldsymbol{B}(\omega)] \mid k_j(\omega) \in \mathbb{R} \text{ and } \operatorname{Im}\omega > 0\}, \tag{3.9}$$

and say that a frequency $\omega \in \mathbb{C}$ is an unstable frequency if $\operatorname{Im}\omega > 0$, and for at least one $j$, $\mathrm{i}k_j(\omega) \in \Sigma_u[\boldsymbol{B}(\omega)]$. It is assumed that the initial-value problem is well posed and that unstable bounded perturbations grow at most exponentially in time; that is, there exists a positive constant $\sigma_0$ such that $\operatorname{Im}\omega < \sigma_0$ for all unstable frequencies. Further, we assume that for $\omega$ with $\operatorname{Im}\omega > \sigma_0$ the spectrum $\Sigma[\boldsymbol{B}(\omega)]$ is bounded, uniformly in $\omega$, away from the imaginary $k$-axis.

**Definition.** The $L$-periodic basic state $W(x)$ is called linearly unstable if there exists an $\omega \in \mathbb{C}$ with $\operatorname{Im}\omega > 0$ such that the unstable spectrum $\Sigma_u[\boldsymbol{B}(\omega)]$ is not empty. Conversely, if for every $\omega \in \mathbb{C}$ with $\operatorname{Im}\omega > 0$ the set $\Sigma_u[\boldsymbol{B}(\omega)]$ is empty we say that the periodic flow $W(x)$ is linearly stable.

This definition is analogous to the corresponding definition in the homogeneous case and follows from the normal mode theory. To be precise one can also include algebraically growing modes. For example, it was found by Brevdo (1988) theoretically that localized perturbations in a homogeneous flow with stable normal modes can under certain conditions grow algebraically in time. However, a local degeneracy of the dispersion relation is necessary for this case to occur. Therefore, we still prefer to use the above definition in order to set up a framework for discussion of exponentially unstable waves. The case of a flow with stable normal modes and algebraically growing localized pulses can be then treated as an exception.

The Floquet decomposition (3.3) is used to transform the boundary-value problem (2.8)–(2.10) into a problem with constant coefficients. We make the substitution

$$Z(x,\omega) = \boldsymbol{Q}(x,\omega)v(x,\omega) \tag{3.10}$$

in (2.8), where $v(x,\omega)$ is a vector function, and by using (3.1) obtain

$$\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}x}v+\boldsymbol{Q}\frac{\mathrm{d}v}{\mathrm{d}x}=(\boldsymbol{A}+\mathrm{i}\omega\boldsymbol{M})\boldsymbol{Q}v+g=[(\boldsymbol{A}+\mathrm{i}\omega\boldsymbol{M})\boldsymbol{Q}\mathrm{e}^{\boldsymbol{B}x}]\mathrm{e}^{-\boldsymbol{B}x}v+g$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}[\boldsymbol{Q}\mathrm{e}^{\boldsymbol{B}x}]\mathrm{e}^{-\boldsymbol{B}x}v+g=\frac{\mathrm{d}\boldsymbol{Q}}{\mathrm{d}x}v+\boldsymbol{QB}v+g. \quad (3.11)$$

Therefore, $v(x,\omega)$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}v(x,\omega)-\boldsymbol{B}(\omega)v(x,\omega)=\boldsymbol{Q}^{-1}(x,\omega)g(x,\omega). \quad (3.12)$$

The assumptions made about $G(x,t)$, $Z_0(x)$ and $Z(x,\omega)$ in (2.4)–(2.6), (2.10), and the boundedness of the norm $\|\boldsymbol{Q}^{-1}(x,\omega)\|$, for a fixed $\omega$ uniformly in $x$, allow us to operate on the system (3.12) with the Fourier transform in $x$,

$$F\{w\}(k)=\widehat{w}(k)=\int_{-\infty}^{\infty}w(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x, \quad w(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{w}(k)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k. \quad (3.13)$$

This gives the equation

$$[\mathrm{i}k\boldsymbol{I}_N-\boldsymbol{B}(\omega)]v(k,\omega)=\int_{-\infty}^{\infty}\boldsymbol{Q}^{-1}(x,\omega)g(x,\omega)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x\equiv T(k,\omega), \quad (3.14)$$

where the hat was dropped for convenience. Since $g(x,\omega)$ in (2.9) has compact support in $x$, the function $T(k,\omega)$ decays faster than $|k|^{-n}$, for any $n>0$, when $k\to\pm\infty$, with $\operatorname{Im}k=0$. The eigenvalues $\mathrm{i}k_j(\omega)$, $1\leqslant j\leqslant N$, of the matrix $\boldsymbol{B}(\omega)$ are bounded, uniformly, away from the imaginary $k$-axis for all $\omega$ with $\operatorname{Im}\omega>\sigma_0$. Therefore, there exists a positive constant $c_0$ such that $|\operatorname{Im}k_j(\omega)|>c_0$, for $1\leqslant j\leqslant N$, uniformly for all such $\omega$. Consequently, for all real $k$ and all $\omega$ with $\operatorname{Im}\omega>\sigma_0$ the eigenvalues $[\mathrm{i}k-\mathrm{i}k_j(\omega)]^{-1}$, $1\leqslant j\leqslant N$, of the matrix $[\mathrm{i}k\boldsymbol{I}_N-\boldsymbol{B}(\omega)]^{-1}$ are uniformly bounded, because $|k-k_j(\omega)|>c_0$ holds. We can, therefore, apply the inversion formula to (3.14) and write

$$v(x,\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[\mathrm{i}k\boldsymbol{I}_N-\boldsymbol{B}(\omega)]^{-1}T(k,\omega)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k. \quad (3.15)$$

The integral in (3.15) converges uniformly for all $\omega$ with $\operatorname{Im}\omega>\sigma_0$. Clearly, the solution $Z(x,\omega)$, (3.10), with $v(x,\omega)$ given by (3.15), satisfies the conditions (2.10). Moreover, for $\omega$ with $\operatorname{Im}\omega>\sigma_0$, $v(x,\omega)\in C^{\infty}(\mathbb{R},\mathbb{R}^N)\bigcap L^1(\mathbb{R},\mathbb{R}^N)$ in $x$, and the application of the Fourier transform to (3.12) is justified.

The solution of the initial-boundary-value problem (2.4)–(2.6) can, therefore, be formally expressed as

$$Z(x,t)=\frac{1}{4\pi^2}\int_{\mathrm{i}\sigma-\infty}^{\mathrm{i}\sigma+\infty}\boldsymbol{Q}(x,\omega)\int_{-\infty}^{\infty}[\mathrm{i}k\boldsymbol{I}_N-\boldsymbol{B}(\omega)]^{-1}T(k,\omega)\mathrm{e}^{\mathrm{i}(kx-\omega t)}\,\mathrm{d}k\mathrm{d}\omega, \quad (3.16)$$

with $\sigma>\sigma_0$. Here it is assumed that the inverse Laplace integral along the Bromwich contour $\{\omega\in\mathbb{C}|\operatorname{Im}\omega=\sigma,\ -\infty<\operatorname{Re}\omega<\infty\}$ exists and is convergent for all $\sigma>\sigma_0$. This assumption must be validated in each particular application of the theory.

## 4. Long-time asymptotics of waves in periodic media

We are interested in the asymptotic behaviour of the solution (3.16) along the ray $x=x_0+Vt$, $x_0=\text{const.}$, for $t\to\infty$. The asymptotics of $Z(x_0,t)$ and $Z(x_0+Vt,t)$,




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when $t \to \infty$, can be treated in the same fashion because it holds that

$$Z(x_0 + Vt, t) = \frac{1}{4\pi^2} \int_{\mathrm{i}\sigma-\infty}^{\mathrm{i}\sigma+\infty} \boldsymbol{Q}(x_0 + Vt, \omega + kV) \times \int_{-\infty}^{\infty} [\mathrm{i}k\boldsymbol{I}_N - \boldsymbol{B}(\omega + kV)]^{-1} T(k, \omega + kV) \mathrm{e}^{\mathrm{i}(kx_0 - \omega t)} \, \mathrm{d}k \mathrm{d}\omega. \quad (4.1)$$

The integral in (4.1) is obtained from the integral in (3.16) by substituting $x = x_0 + Vt$ and performing the change of variables $k \to k$, $\omega - kV \to \omega$. We treat, therefore, only the solution in the form (3.16) and write it as

$$Z(x_0, t) = \frac{1}{4\pi^2} \int_{\mathrm{i}\sigma-\infty}^{\mathrm{i}\sigma+\infty} \boldsymbol{Q}(x_0, \omega) R(x_0, \omega) \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}\omega \quad (4.2)$$

with

$$R(x_0, \omega) = \int_{-\infty}^{\infty} [\mathrm{i}k\boldsymbol{I}_N - \boldsymbol{B}(\omega)]^{-1} T(k, \omega) \mathrm{e}^{\mathrm{i}kx_0} \, \mathrm{d}k. \quad (4.3)$$

The representation (4.2), (4.3) is a generalization to the spatially periodic case of the treatment given by Briggs (1964) in the spatially homogeneous case.

According to the definition of $\sigma$ in (3.16) the vector function $R(x_0, \omega)$ is analytic for $\omega$ with $\operatorname{Im} \omega > \sigma_0$. When this function can be analytically continued all the way down to the real $\omega$-axis, i.e. in the strip

$$\mathcal{S}_\sigma = \{\omega \in \mathbb{C} \,|\, 0 \leqslant \operatorname{Im}\, \omega \leqslant \sigma, \ -\infty < \operatorname{Re} \omega < \infty\}, \quad (4.4)$$

and the resulting integrand in (4.2) decays fast enough for $\operatorname{Re} \omega \to \pm\infty$ in this strip, then the integral in (4.2) is equal to

$$Z(x_0, t) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \boldsymbol{Q}(x_0, \omega) \bar{R}(x_0, \omega) \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}\omega, \quad (4.5)$$

where $\bar{R}(x_0, \omega)$ denotes the continued function. The Riemann–Lebesgue lemma (see Körner 1988, §52) applied to the integral in (4.5) gives $Z(x_0, t) \to 0$ for $t \to \infty$. Thus the central point in the analysis is to determine the singularities of the function $\bar{R}(x_0, \omega)$ in the strip $\omega \in \mathcal{S}_\sigma$. This is done by analytically continuing the function (4.3) from above the Bromwich contour down along the vertical line $\{\omega \in \mathbb{C} | \operatorname{Re} \omega = \omega_\mathrm{r} = \text{const.}\}$, for each $\omega_\mathrm{r} \in \mathbb{R}$ and determining the circumstances under which this continuation fails. We denote a movement of $\omega$ from above the Bromwich contour down along such a vertical line by $\omega \searrow$, so that $\omega \searrow \omega_0$ denotes such a movement along the vertical line passing through the point $\omega_0$; that is $\omega = \operatorname{Re} \omega_0 + \mathrm{i}y$ with $\operatorname{Im} \omega_0 \leqslant y \leqslant \sigma + \varepsilon$, where $\varepsilon > 0$ is any positive real number.

The analytic continuation of the function $R(x_0, \omega)$ can fail at a point $\omega_0$, when $\omega \searrow \omega_0$, if and only if two or more numbers $k_j(\omega)$, where $\mathrm{i}k_j(\omega)$, $1 \leqslant j \leqslant N$, are the eigenvalues of the operator $\boldsymbol{B}(\omega)$, 'pinch' the deformed contour of integration in (4.3) during their movement in the $k$-plane, see figure 2.

This means that at least two of the colliding numbers $k_j(\omega)$ must originate on opposite sides of the real $k$-axis. In other words, at least two numbers $k_j(\omega)$ that collide for $\omega = \omega_0$ at $k_0 = k_j(\omega_0)$ (i.e. $\mathrm{i}k_0$ is a multiple eigenvalue of $\boldsymbol{B}(\omega)$ at $\omega = \omega_0$) move to opposite sides of the real $k$-axis and stay there in the limit $\operatorname{Im}(\omega - \omega_0) \to \infty$ while $\operatorname{Re}(\omega - \omega_0) = 0$. A classification of the singularities of the function $R(x_0, \omega)$ at the points of pinching can be carried out as in the homogeneous case (cf. Brevdo

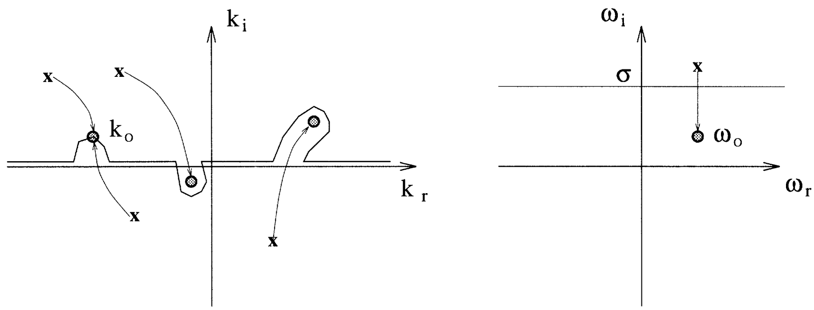


Figure 2. Movement of the wave numbers $k_j(\omega)$ as $\omega \searrow \omega_0$ resulting in pinching of the deformed Fourier contour.

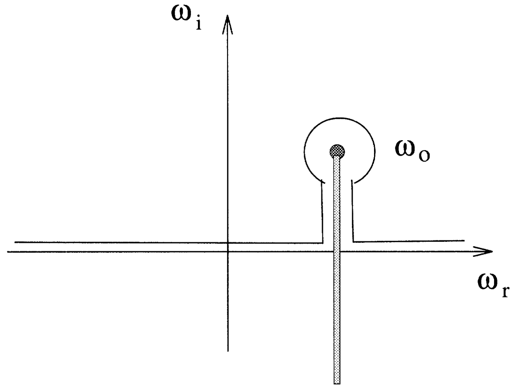


Figure 3. Bromwich contour deformed around a branch point.

1988). The singularity at any such point $\omega_0$ is of the form

$$\frac{1}{(\omega - \omega_0)^s}, \tag{4.6}$$

where $s$ is a positive rational number which depends on the number and multiplicity of the colliding roots in $k$ of the determinant

$$D(k, \omega) = \det[\mathrm{i}k\boldsymbol{I}_N - \boldsymbol{B}(\omega)]. \tag{4.7}$$

When $s$ is an integer then the function $R(x_0, \omega)$ has a pole at $\omega_0$ and when $s$ is not integer-valued a branch cut of $R(x_0, \omega)$ is chosen to be the vertical semi-line with its upper end at $\omega_0$, i.e. $\{\omega \in \mathbb{C} | \operatorname{Re}\omega = \operatorname{Re}\omega_0,\ \operatorname{Im}\omega \leqslant \operatorname{Im}\omega_0\}$. Assuming that there is only a finite number of such singularities, which is reasonable for a great variety of applications, the Bromwich contour can be deformed as is shown in figure 3.

In order for the deformation to make sense, the integral along the original Bromwich contour should equal the integral along the deformed contour. To validate this assumption an estimate of the decay of the norm $\|\boldsymbol{Q}(x_0, \omega)\bar{R}(x_0, \omega)\|$ is required as $\omega$ tends to infinity in both directions in the strip $\mathcal{S}_\sigma$. It is difficult to prove such an estimate in general, but is a reasonable hypothesis, and in the examples (cf. § 6) it is shown to be valid.

The Riemann–Lebesgue lemma can be applied to the integral along the deformed Bromwich contour showing that the non-decaying contributions to the asymptotics of the solution (3.16), for $t \to \infty$, come only from the singular points with $\operatorname{Im}\omega \geqslant 0$. The singular point $(k, \omega)$ with maximal $\operatorname{Im}\omega$ makes the dominant contribution. Let




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this point be $(k_0, \omega_0)$. Then the dominant contribution to the asymptotics of $Z(x_0, t)$ is of the form

$$Z(x_0,t) \sim CT(k_0,\omega_0) = \boldsymbol{Q}(x_0,\omega_0)P(k_0,\omega_0)\mathrm{e}^{\mathrm{i}k_0x_0}t^{s-1}\mathrm{e}^{-\mathrm{i}\omega_0 t}. \tag{4.8}$$

Here the vector $P(k_0, \omega_0)$ depends only on $k_0$ and $\omega_0$ through the dependence of the singularity of the function $\bar{R}(x_0, \omega)$ at $\omega_0$ on $[\mathrm{i}k\boldsymbol{I}_N - \boldsymbol{B}(\omega)]^{-1}T(k, \omega)$. The exponent $s-1$ of the algebraic factor $t^{s-1}$ in (4.8), with $s$ as in (4.6), is positive in case $s > 1$. So theoretically a variety of algebraically growing responses is possible when a contributing $\omega_0$ with $\operatorname{Im}\omega_0 = 0$ corresponds to $s > 1$ (cf. Brevdo (1988) for the argument in the spatially homogeneous case and for the practical implementation of the collision criterion in such a case).

At the contributing point $(k_0, \omega_0)$ the dispersion relation function $D(k, \omega)$ has a multiple root in $k$. When this root is a double root it satisfies

$$D(k_0,\omega_0) = 0, \quad D_k(k_0,\omega_0) = 0. \tag{4.9}$$

If $D_\omega(k_0, \omega_0) \neq 0$ the equation $D(k, \omega) = 0$ has a single solution in $\omega$ in some neighbourhood of $(k_0, \omega_0)$. We denote this solution by $\omega = \omega(k)$. At $k = k_0$, $\omega'(k_0) = 0$ holds, i.e. the function $\omega = \omega(k)$ has a saddle point at $k_0$. This is a weak necessary condition for absolute instability, since it does not contain the crucial information concerning the origin of the colliding roots. For multiple colliding roots the theory in Brevdo (1988) can be generalized directly to the spatially periodic case. We do not present it here.

The function $D(k, \omega)$ plays a role analogous to that played by the dispersion relation function in the homogeneous case. In fact, a substitution of the expression

$$Z(x,t) = \boldsymbol{Q}(x,\omega)h\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \tag{4.10}$$

where $h$ is a constant vector, into the homogeneous equation associated with (2.4) is an operation directly analogous to the normal mode substitution in the homogeneous case and gives the equation

$$[\mathrm{i}k\boldsymbol{I}_N - \boldsymbol{B}(\omega)]h\mathrm{e}^{\mathrm{i}(kx-\omega t)} = 0. \tag{4.11}$$

Hence, a non-trivial solution of the form (4.10) exists if and only if $k$ and $\omega$ satisfy

$$D(k,\omega) = \det[\mathrm{i}k\boldsymbol{I}_N - \boldsymbol{B}(\omega)] = 0, \tag{4.12}$$

and the analogy with the homogeneous case is complete.

A pinching of the deformed Fourier contour takes place for $\omega = \omega_0$ and $k = k_0$ if and only if there is a pinching of the deformed imaginary axis in the complex $k$-plane by the eigenvalues of the matrix $\boldsymbol{B}(\omega)$, that occurs for the same $\omega = \omega_0$ and for $k = \mathrm{i}k_0$. According to (3.4) this can take place if and only if at least two Floquet multipliers $\mu$ of the monodromy matrix $\boldsymbol{\Phi}(L, \omega)$, that is the roots $\mu_j(\omega)$, $1 \leqslant j \leqslant N$, in $\mu$ of

$$\Delta(\mu,\omega) = \det[\boldsymbol{\Phi}(L,\omega) - \mu\boldsymbol{I}_N] = 0, \tag{4.13}$$

originating on opposite sides of the unit circle $\{|\mu| = 1\}$ in the complex $\mu$-plane collide at the point $\mu_0 = \mathrm{e}^{\mathrm{i}k_0L}$, when $\omega \searrow \omega_0$.

It is important to note that the dominant term of the asymptotics of the solution (4.1) along the ray $x = x_0 + Vt$, $x_0 = \text{const.}$, for $t \to \infty$, takes the form

$$\begin{aligned} Z(x_0+Vt,t) &\sim CT(k_0,\omega_0) \\ &= \boldsymbol{Q}(x_0+Vt,\omega_0+k_0V)P(k_0,\omega_0+k_0V)\mathrm{e}^{\mathrm{i}k_0x_0}t^{s-1}\mathrm{e}^{-\mathrm{i}\omega_0 t}, \end{aligned} \tag{4.14}$$

where $P(k_0, \omega_0 + k_0 V)$ is a vector independent of $x$ and $t$. Since $\boldsymbol{Q}(x, \omega)$ is $L$-periodic in $x$, this term grows in time as

$$t^{s-1} e^{\operatorname{Im} \omega_0 t} \tag{4.15}$$

and oscillates as

$$\boldsymbol{Q}(x_0 + Vt, \omega_0 + k_0 V) e^{-i \operatorname{Re} \omega_0 t}. \tag{4.16}$$

If, for $V \neq 0$, the real frequency $\operatorname{Re} \omega_0$ associated with the Floquet multiplier $\mu_0$ is non-zero and the ratio $(L/2\pi V) \operatorname{Re} \omega_0$ is not a rational number then the oscillatory part of the asymptotics is, to the leading order, quasi-periodic in time. The quasi-periodic response is in contrast to the asymptotics (4.8) in the frame in which the basic state is stationary, and to the asymptotics in the homogeneous case where the oscillatory part is periodic in every reference frame.

## 5. The signalling problem in periodic media

A problem of interest in applications is the asymptotics in space and time of a flow subjected to a spatially localized forcing that is periodic in time. This is the so-called linear signalling problem. In this section we generalize this problem to the case where the basic flow or medium is spatially periodic. Its formulation in an unstable case is meaningful only if the flow is absolutely stable because in the absolutely unstable case any linear structure that might tend to evolve will eventually be destroyed by the perturbations that grow at every point in space. In the absolutely stable but convectively unstable case we are interested in the asymptotics of the solution $Z(x, t)$ of (2.4)–(2.6), for $x \to \pm\infty$, $t \to \infty$ when the forcing term $G(x, t)$ in (2.4) is a periodic function of $t$ with compact stationary support in a spatial coordinate relative to a reference frame moving at speed $c$. A periodic solution of a nonlinear equation, e.g. Ginzburg–Landau equation, is often a travelling wave which is stationary only in a moving frame of reference. A periodic forcing, whose spatial support is at rest in the absolute frame of reference, is moving relative to the frame at which such a solution is stationary. Hence, a signalling problem with a moving forcing can be of practical importance for periodic waves. We treat the problem of a moving forcing based on the analysis for a forcing with stationary support. The basic state $W(x)$, (2.2), in this section is taken to be absolutely stable, but convectively unstable.

### (a) *Forcing with stationary support*

The forcing in (2.4) is set to be

$$G(x, t) = f(x - x_0) e^{-i\omega_0 t}, \quad \text{with} \quad f(x) \in C_0^\infty(\mathbb{R}, \mathbb{R}^N) \quad \text{and} \quad \operatorname{Im} \omega_0 = 0. \tag{5.1}$$

Since the flow is absolutely stable, a non-vanishing initial condition $Z_0(x)$ in (2.5) can trigger only a diminishing disturbance at every point in space, so its influence will be ignored, i.e. $Z_0(x) = 0$. The boundary condition (2.6) is replaced below by an assumption on the decay in space of the Laplace transform of $Z(x, t)$. For every fixed $x$ the function $Z(x, t)$ is assumed to be bounded in $t$.

Application of the Laplace transform (2.7) to the equation

$$\boldsymbol{M} Z_t + Z_x = \boldsymbol{A}(x) Z + f(x - x_0) e^{-i\omega_0 t}, \quad \text{with} \quad x \in \mathbb{R} \quad \text{and} \quad t > 0, \tag{5.2}$$

results, after the substitution (3.10), in the equation

$$\frac{d}{dx} v(x, \omega) - \boldsymbol{B}(\omega) v(x, \omega) = i \boldsymbol{Q}^{-1}(x, \omega) f(x - x_0) \frac{1}{\omega - \omega_0}. \tag{5.3}$$

Assuming that for every fixed $\omega$ on the Bromwich contour the function $v(x,\omega)$ satisfies the boundary conditions

$$\lim_{x\to\pm\infty} \|v(x,\omega)\| = 0, \tag{5.4}$$

and $v(x,\omega) \in C^\infty(\mathbb{R},\mathbb{R}^N)\bigcap L^1(\mathbb{R},\mathbb{R}^N)$ in $x$, we solve (5.3) by applying the Fourier transform:

$$v(x,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega-\omega_0}[\mathrm{i}k\boldsymbol{I}_N - \boldsymbol{B}(\omega)]^{-1}T_1(k,\omega;x_0)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k, \tag{5.5}$$

where

$$T_1(k,\omega;x_0) = \int_{-\infty}^{\infty} \mathrm{i}\boldsymbol{Q}^{-1}(x,\omega)f(x-x_0)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x. \tag{5.6}$$

It is easy to see that $v(x,\omega)$ given by (5.5) satisfies the required conditions. Indeed, $T_1(k,\omega;x_0)$ is a rapidly decaying function of $k$ at infinity on the real $k$-axis, because $f(x) \in C_0^\infty(\mathbb{R},\mathbb{R}^N)$, and the norm $\|[\mathrm{i}k\boldsymbol{I}_N - \boldsymbol{B}(\omega)]^{-1}\|$ is bounded from above uniformly in real $k$ for every fixed $\omega$ on the Bromwich contour, as shown previously.

The solution (3.16) now formally reads

$$Z(x,t) = \frac{1}{4\pi^2}\int_{\mathrm{i}\sigma-\infty}^{\mathrm{i}\sigma+\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega-\omega_0}\boldsymbol{Q}(x,\omega)\int_{-\infty}^{\infty}[\mathrm{i}k\boldsymbol{I}_N - \boldsymbol{B}(\omega)]^{-1}T_1(k,\omega;x_0)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k\mathrm{d}\omega. \tag{5.7}$$

It is a generalization to base flows that are spatially periodic of the treatment by Briggs (1964) for the homogeneous case. Consequently, the causality condition and the spatial asymptotics can be obtained in a similar fashion. The causality condition is: spatially growing contributions to the asymptotics of the solution (5.7) for $x\to\infty$, $t\to\infty$ come only from the roots in $k$, $k_j(\omega)$, $1\leqslant j\leqslant N$, of the dispersion relation function (4.7) that cross from the upper half of the complex $k$-plane, i.e. $\operatorname{Im} k > 0$, to its lower half, i.e. $\operatorname{Im} k < 0$, when $\omega$ moves from above the Bromwich contour down to the real excitation frequency $\omega_0$, i.e. $\omega\searrow\omega_0$. Contributions for $x\to-\infty$, $t\to\infty$ come from the roots that cross from below the real $k$-axis to the upper half plane. This condition assumes the analytic continuation of the corresponding function and the deformation of the contour in the outer integral in (5.7), as in § 4, which has to be justified in every particular case. Given the matrix $\boldsymbol{B}(\omega)$, either analytically or numerically, this condition can be implemented by following the movements of the images of the contour $\{\omega \mid \operatorname{Im}\omega = \sigma\}$ under the transformations $k = k_j(\omega)$, $1\leqslant j\leqslant N$, in the $k$-plane, when $\sigma\to 0_+$, as it was done in Brevdo (1992) for homogeneous plane Poiseuille flow.

Similar to the pinching condition in the treatment of the absolute and convective instabilities, the causality condition can be formulated in terms of the Floquet multipliers $\mu(\omega)$ of the monodromy matrix $\boldsymbol{\Phi}(L,\omega)$ in the following manner. Spatially growing contributions to the asymptotics of the solution (5.7) for $x\to\infty$, $t\to\infty$ come only from the Floquet multipliers $\mu_j(\omega)$, $1\leqslant j\leqslant N$, of the monodromy matrix $\boldsymbol{\Phi}(L,\omega)$, i.e. the roots in $\mu$ of (4.13), that cross from the interior of the unit circle in the complex $\mu$-plane, i.e. $\{|\mu|<1\}$, to its exterior, i.e. $\{|\mu|>1\}$, when $\omega$ moves from above the Bromwich contour down to the real excitation frequency $\omega_0$, i.e. $\omega\searrow\omega_0$. Contributions for $x\to-\infty$, $t\to\infty$ come from the Floquet multipliers that cross from the exterior to the interior of the circle.

We will demonstrate the usefulness of this simplified version of the causality condition in § 7. A practical implementation of this condition as well as of the pinching





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condition given in terms of the Floquet multipliers in § 4 is similar to that of the corresponding conditions formulated in terms of the eigenvalues of the matrix $\boldsymbol{B}(\omega)$. However, it is certainly advantageous to use the Floquet multipliers when $\boldsymbol{B}(\omega)$ cannot be computed exactly, as in most cases of flows with shear.

Let, for $x \to \infty$, $k = k_m(\omega)$ denote the crossing root with maximum negative imaginary part among all crossing roots at the end of their trajectories, that is with maximum $-\operatorname{Im} k_m(\omega_0)$. Let $k_0 = k_m(\omega_0)$. Then the dominant contribution to the spatial asymptotics of the solution (5.7) for $x \to \infty$, $t \to \infty$ is given by

$$Z(x,t) \sim CS(k_0,\omega_0) = \boldsymbol{Q}(x,\omega_0)P(k_0,\omega_0,x_0)\mathrm{e}^{-\mathrm{i}\omega_0 t}\mathrm{e}^{-\operatorname{Im} k_0 x}\mathrm{e}^{\mathrm{i}\operatorname{Re} k_0 x}. \tag{5.8}$$

The vector $P(k_0,\omega_0,x_0)$ does not depend on $x$ and $t$. For $x \to -\infty$, $t \to \infty$ the dominant contribution of the same form comes from the crossing root with maximum positive imaginary part. It is noteworthy that the undulating part of the growing contribution (5.8) is quasi-periodic in space if $\operatorname{Re} k_0$ is non-vanishing and the ratio $\frac{L}{2\pi}\operatorname{Re} k_0$ is not a rational number. In the homogeneous case the undulating part of the asymptotics to the leading order is always periodic. The temporal behaviour of the entire response is *ca.* $\mathrm{e}^{-\mathrm{i}\omega_0 t}$. In the case of a moving simple harmonic forcing the response can be quasi-periodic in time.

### (*b*) *Moving forcing*

In this case the forcing (2.4) is assumed to have stationary spatial support relative to a frame moving at speed $c$. It is of the form

$$G(x,t) = f(x - x_0 - ct)\mathrm{e}^{-\mathrm{i}\omega_0 t}, \quad \text{with} \quad f(x) \in C_0^\infty(\mathbb{R},\mathbb{R}^N) \quad \text{and} \quad \operatorname{Im}\omega_0 = 0. \tag{5.9}$$

The analysis proceeds as in the preceding subsection. Equation (5.2) reads for this forcing

$$\boldsymbol{M}Z_t + Z_x = \boldsymbol{A}(x)Z + f(x - x_0 - ct)\mathrm{e}^{-\mathrm{i}\omega_0 t}, \quad \text{with} \quad x \in \mathbb{R} \quad \text{and} \quad t > 0, \tag{5.10}$$

and equation (5.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}v(x,\omega) - \boldsymbol{B}(\omega)v(x,\omega) = \boldsymbol{Q}^{-1}(x,\omega)\int_0^\infty f(x - x_0 - ct)\mathrm{e}^{\mathrm{i}(\omega-\omega_0)t}\,\mathrm{d}t. \tag{5.11}$$

The integral on the right-hand side of (5.11) cannot in general be computed explicitly as in the case when $c = 0$. The solution $v(x,\omega)$ now has to be represented as a triple integral, i.e. Laplace–Fourier-inverse Fourier integral. It reads

$$v(x,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\mathrm{i}k\boldsymbol{I}_N - \boldsymbol{B}(\omega)]^{-1} \times \left[\int_{-\infty}^{\infty}\boldsymbol{Q}^{-1}(s,\omega)\int_0^\infty f(s - x_0 - ct)\mathrm{e}^{\mathrm{i}(\omega-\omega_0)t}\,\mathrm{d}t\,\mathrm{e}^{-\mathrm{i}ks}\,\mathrm{d}s\right]\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k. \tag{5.12}$$

We introduce a change of variable, $\zeta = s - ct$ in the integral in (5.12) and obtain for the expression in the brackets

$$\begin{aligned}\mathcal{T} &= \int_{-\infty}^{\infty}\boldsymbol{Q}^{-1}(s,\omega)\int_0^\infty f(s - x_0 - ct)\mathrm{e}^{\mathrm{i}(\omega-\omega_0)t}\,\mathrm{d}t\,\mathrm{e}^{-\mathrm{i}ks}\,\mathrm{d}s \\ &= \int_{-\infty}^{\infty}\int_0^\infty \boldsymbol{Q}^{-1}(\zeta + ct,\omega)f(\zeta - x_0)\mathrm{e}^{\mathrm{i}(\omega-\omega_0-kc)t}\mathrm{e}^{-\mathrm{i}k\zeta}\,\mathrm{d}t\mathrm{d}\zeta \\ &= \int_{-\infty}^{\infty}\int_0^\infty \boldsymbol{Q}^{-1}(\zeta + ct,\omega)\mathrm{e}^{\mathrm{i}(\omega-\omega_0-kc)t}\,\mathrm{d}t\,f(\zeta - x_0)\mathrm{e}^{-\mathrm{i}k\zeta}\,\mathrm{d}\zeta. \end{aligned} \tag{5.13}$$





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The convergence of the integral in (5.13) can be easily verified for $\omega$ with $\operatorname{Im}\omega > 0$.

The matrix function $\boldsymbol{Q}^{-1}(x,\omega)$ is $C^{\infty}$ and $L$-periodic in $x$ for every fixed $\omega \in \mathbb{C}$ and, therefore, has a convergent Fourier series representation given by

$$\boldsymbol{Q}^{-1}(x,\omega) = \sum_{n=-\infty}^{\infty} \boldsymbol{Q}_n(\omega)\mathrm{e}^{\mathrm{i}2\pi nx/L}. \tag{5.14}$$

The above series converge absolutely for every fixed $\omega \in \mathbb{C}$. We assume that the convergence is uniform in the strip $\mathcal{S}_\sigma$ defined in (4.4). The $\boldsymbol{Q}_n(\omega)$ are entire functions of $\omega$ for all integers $n \in \mathbb{Z}$, since $\boldsymbol{Q}^{-1}(x,\omega)$ is an entire function of $\omega$. The expression $\mathcal{T}$ in (5.13) can be evaluated by substituting (5.14) and integrating with respect to $t$ to find

$$\mathcal{T} = \sum_{n=-\infty}^{\infty} \boldsymbol{Q}_n(\omega) f\left(k - \frac{2\pi n}{L}\right) \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}(k-2\pi n/L)x_0}}{\omega - \omega_0 - kc + (2\pi n/L)c}, \tag{5.15}$$

where $f(k-2\pi n/L)$ is the Fourier transform of $f(x)\mathrm{e}^{\mathrm{i}(2\pi n/L)x}$. We substitute (5.15) in place of the expression in the brackets in (5.12), apply an inverse Laplace transform to $\boldsymbol{Q}(x,\omega)v(x,\omega)$ (see (3.10)) and, after interchanging the summation with the inverse Laplace integral and rearranging terms, obtain

$$Z(x,t) = \frac{\mathrm{i}}{4\pi^2} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(2\pi n/L)x_0} \int_{\mathrm{i}\sigma-\infty}^{\mathrm{i}\sigma+\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega - \omega_0 - kc + (2\pi n/L)c} \boldsymbol{Q}(x,\omega) \times \int_{-\infty}^{\infty} [\mathrm{i}k\boldsymbol{I}_N - \boldsymbol{B}(\omega)]^{-1} \boldsymbol{Q}_n(\omega) f(k - 2\pi n/L)\mathrm{e}^{\mathrm{i}k(x-x_0)}\,\mathrm{d}k\mathrm{d}\omega. \tag{5.16}$$

A change of variables $k \to k$, $\omega - kc \to \omega$ in the double integral in (5.16) gives

$$Z(x,t) = \frac{\mathrm{i}}{4\pi^2} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(2\pi n/L)x_0} \int_{i\sigma-\infty}^{\mathrm{i}\sigma+\infty} \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega - \omega_0 + (2\pi n/L)c} \int_{-\infty}^{\infty} \boldsymbol{Q}(x,\omega+kc) \times [\mathrm{i}k\boldsymbol{I}_N - \boldsymbol{B}(\omega+kc)]^{-1} \boldsymbol{Q}_n(\omega+kc) f(k-2\pi n/L) \times \mathrm{e}^{\mathrm{i}k(x-x_0-ct)}\,\mathrm{d}k\mathrm{d}\omega. \tag{5.17}$$

The double inverse Fourier inverse Laplace integral in (5.17) has the form analogous to that of the integral in (5.7). Its asymptotic evaluation, when $(x-x_0) \to \pm\infty$, $t \to \infty$ is similar. For a real frequency of excitation $\omega_0$, the growing contributions to the spatial asymptotics of the solution $Z(x,t)$, when $(x - x_0) \to \infty$, $t \to \infty$ come from the roots in $k$ of

$$D(k, \omega + kc) = \det[\mathrm{i}k\boldsymbol{I}_N - \boldsymbol{B}(\omega + kc)] \tag{5.18}$$

that cross from above to below the real $k$-axis, when $\omega$ is brought from above the Bromwich contour down to $(\omega_0 - (2\pi n/L)c)$ for each $n \in \mathbb{Z}$.

Let $n \in \mathbb{Z}$ be fixed. For $x \to \infty$, we denote by $k = k_{nm}(\omega)$ the crossing root with maximum negative imaginary part among all the crossing roots at the end of their trajectories, that cross the real $k$-axis for $\omega \searrow (\omega_0 - (2\pi n/L)c)$. This root has maximum $-\operatorname{Im} k_{nm}(\omega_0 - (2\pi n/L)c)$ among all such roots for a fixed $n \in \mathbb{Z}$. Let $k_{n0} = k_{nm}(\omega_0 - (2\pi n/L)c)$. Then the dominant contribution to the spatial asymptotics of the solution (5.17) for $(x - x_0) \to \infty$, $t \to \infty$ in which all possible frequencies of the response are represented, is given by

$$Z(x,t) \sim CS = \sum_{n=-\infty}^{\infty} \boldsymbol{Q}(x, \omega_0 - (2\pi n/L)c + k_{n0}c) P_n(k_{n0}, \omega_0 - (2\pi n/L)c, c, x_0) \times \mathrm{e}^{-\mathrm{i}(\omega_0 - (2\pi n/L)c + k_{n0}c)t} \mathrm{e}^{\mathrm{i}k_{n0}(x - x_0)}. \tag{5.19}$$

The vectors $P_n(k_{n0}, \omega_0 - (2\pi n/L)c, c, x_0)$, $n \in \mathbb{Z}$, do not depend on $x$ and $t$. For $x \to -\infty$, $t \to \infty$ the dominant contribution of the same form comes from the crossing roots with maximum positive imaginary parts. A formulation of the crossing condition in terms of the Floquet multipliers $\mu$ does not seem to be of a practical advantage in the present case, since in such a formulation both $\mu$ and the function $\ln \mu$ appear in the equation

$$\det[\boldsymbol{\Phi}(L, \omega - \mathrm{i}c \ln \mu) - \mu \boldsymbol{I}_N] = 0. \tag{5.20}$$

Recall that the forcing is moving at speed $c$ relative to the frame at which the basic state is stationary. In practice, a spatial response in the frame of reference at which the forcing has a stationary finite support is of interest. Relative to this frame the basic state is moving at speed $-c$. The coordinate $\eta$ in this frame is related to $x$ through

$$x = \eta + ct. \tag{5.21}$$

We substitute (5.21) into (5.19) and obtain

$$Z(\eta + ct, t) \sim CS = \sum_{n=-\infty}^{\infty} \boldsymbol{Q}(\eta + ct, \omega_0 - (2\pi n/L)c + k_{n0}c) \times P_n(k_{n0}, \omega_0 - (2\pi n/L)c, c, x_0) \mathrm{e}^{-\mathrm{i}(\omega_0 - (2\pi n/L)c)t} \mathrm{e}^{\mathrm{i}k_{n0}(\eta - x_0)}. \tag{5.22}$$

The response to a moving source, (5.22), when the medium is spatially periodic, is significantly more complicated than the response when the medium is spatially homogeneous. In the homogeneous case all the terms of the infinite series (5.14), and with them of the series (5.19), (5.22), with the exception of the term for $n = 0$, are zero and no space dependent factor $\boldsymbol{Q}$ is present in (5.19), (5.22). Moreover, the entire response in the homogeneous case has a single frequency $\omega_0$, i.e. the frequency of excitation. On the other hand, in the spatially periodic case the $n$th term of the response (5.22) considered as a function of time is a product of $\mathrm{e}^{-\mathrm{i}\omega_0 t}$ with a time-periodic function of period $L/cn$. Hence, both the spatial response of a spatially periodic travelling wave to an oscillatory forcing and its temporal response are in general quasi-periodic.

## 6. Instability of the spatially periodic solutions of the Ginzburg–Landau equation

The Ginzburg–Landau (GL) equation is a model equation for near critical hydrodynamic stability problems (cf. DiPrima *et al.* 1971; Stewartson & Stuart 1971; Stuart & DiPrima 1978; Eckhaus 1993; and references therein). In addition to its importance in applications the GL equation provides an interesting example for the theory presented in §§ 2–4, because the Floquet decomposition, for the GL equation linearized about a finite-amplitude periodic travelling wave, can be constructed explicitly. Moreover, the questions that arise in the analysis of the GL equation are analogous to those for more general systems including the Navier–Stokes equations (cf. § 7).

The complex Ginzburg–Landau (CGL) equation is

$$\frac{\partial \Psi}{\partial t} = r\Psi + \beta_1 \frac{\partial^2 \Psi}{\partial x^2} + \beta_2 |\Psi|^2 \Psi, \qquad x \in \mathbb{R} \quad \text{and} \quad t > 0, \tag{6.1}$$

where $\Psi$ is a complex-valued scalar function, $r$ is a real positive parameter of order unity and $\beta_1, \beta_2 \in \mathbb{C}$ are complex-valued parameters. The equation can be scaled so that $r = 1$ and, in order to consider the supercritical case, $(\beta_1, \beta_2)$ can be scaled so that $\operatorname{Re}\beta_1 = 1$ and $\operatorname{Re}\beta_2 = -1$. After such a scaling the CGL equation reduces to

$$\frac{\partial \Psi}{\partial t} = \Psi + (1 + \mathrm{i}a)\frac{\partial^2 \Psi}{\partial x^2} + (-1 + \mathrm{i}b)|\Psi|^2 \Psi, \tag{6.2}$$

where $a, b \in \mathbb{R}$ are arbitrary real numbers.

There exists a periodic, finite-amplitude, travelling wave state of (6.2) of the form

$$\Psi(x,t) = (\xi_1 + \mathrm{i}\xi_2)\, e^{\mathrm{i}\alpha(x+ct)}, \qquad \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \xi \in \mathbb{R}^2. \tag{6.3}$$

We will use $\xi$ to denote both a vector in $\mathbb{R}^2$ as well as a complex number $\xi = \xi_1 + \mathrm{i}\xi_2 \in \mathbb{C}$. Let

$$\rho = \sqrt{\xi_1^2 + \xi_2^2} = |\xi|. \tag{6.4}$$

Then the parameters in (6.2)–(6.4) satisfy

$$\rho^2 + \alpha^2 = 1 \quad \text{and} \quad c = b/\alpha - (a+b)\alpha. \tag{6.5}$$

We restrict $\alpha$ to $|\alpha| < 1$ to exclude the zero amplitude solution and without loss of generality take $\alpha$ positive. For each fixed $(a, b) \in \mathbb{R}^2$ there is a one-parameter family of spatially periodic travelling waves:

$$(\rho(\ell), \alpha(\ell), c(\ell)) = (\sin \ell, \cos \ell, b/\cos \ell - (a+b)\cos \ell), \quad \ell \in (0, \tfrac{1}{2}\pi). \tag{6.6}$$

The time asymptotics of the CGL equation, linearized about the spatially periodic travelling wave (6.6), will be considered.

The CGL equation is reformulated for application of the theory of §§ 2–4 by defining

$$\Psi = u_1 + \mathrm{i}u_2, \quad (1 + \mathrm{i}a)\Psi_x = v_1 + \mathrm{i}v_2, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{6.7}$$

Then equation (6.2) can be written as

$$\boldsymbol{M} Z_t + Z_x = F(Z), \tag{6.8}$$

where

$$Z = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^4, \quad \boldsymbol{M} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} \\ -\boldsymbol{I}_2 & \boldsymbol{0} \end{pmatrix}, \quad F(Z) = \begin{pmatrix} \boldsymbol{m}^{-1} v \\ -u + (u_1^2 + u_2^2)\boldsymbol{n}u \end{pmatrix}, \tag{6.9}$$

with

$$\boldsymbol{m} = \begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{n} = \begin{pmatrix} 1 & b \\ -b & 1 \end{pmatrix}. \tag{6.10}$$

A basic state, relative to a frame moving at speed $-c$, denoted $W(x)$, is a periodic solution of

$$(\boldsymbol{I}_4 + c\boldsymbol{M})W_x = F(W). \tag{6.11}$$

The basic state (6.3), in terms of the new coordinates, takes the form

$$W(x) = \boldsymbol{Q}(x)\begin{pmatrix} \xi \\ -\alpha \boldsymbol{m}\boldsymbol{J}_2\xi \end{pmatrix}, \quad \text{for} \quad \xi \in \mathbb{R}^2, \tag{6.12}$$

where

$$\boldsymbol{Q}(x) = \begin{pmatrix} \boldsymbol{R}_{\alpha x} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}_{\alpha x} \end{pmatrix}, \quad \boldsymbol{R}_{\alpha x} = \boldsymbol{I}_2 \cos \alpha x - \boldsymbol{J}_2 \sin \alpha x, \quad \boldsymbol{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.13}$$

Note that $x$ here is the coordinate in the moving frame and that $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^{\mathrm{T}}$, i.e. $\boldsymbol{Q}$ is orthogonal. Also $W(x+L) = W(x)$ for $L = 2\pi/\alpha$. The parameters $\rho = |\xi|$, $c$ and $\alpha$ must satisfy (6.5).

The linear stability problem for $W(x)$ is obtained by linearizing (6.8) about the basic state (6.12) in the moving frame resulting in

$$\boldsymbol{M}Z_t + (\boldsymbol{I}_4 + c\boldsymbol{M})Z_x = DF(W(x))Z. \tag{6.14}$$

Since $\boldsymbol{M}^2 = \boldsymbol{0}$ and $(\boldsymbol{I}_4 + c\boldsymbol{M})^{-1} = \boldsymbol{I}_4 - c\boldsymbol{M}$, the linear stability equation can be written as

$$\boldsymbol{M}Z_t + Z_x = \boldsymbol{A}(x)Z + G(x,t), \tag{6.15}$$

with

$$\boldsymbol{A}(x) = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{m}^{-1} \\ -\boldsymbol{I}_2 + \rho^2\boldsymbol{n} + 2\boldsymbol{n}\boldsymbol{R}_{\alpha x}\xi\xi^{\mathrm{T}}\boldsymbol{R}_{\alpha x}^{\mathrm{T}} & c\boldsymbol{m}^{-1} \end{pmatrix}, \tag{6.16}$$

where for completeness a source term $G(x,t)$ is also included on the right-hand side of (6.15).

The Laplace-transformed function $Z(x,\omega)$ (cf. equation (2.7) ) satisfies

$$Z_x(x,\omega) = [\boldsymbol{A}(x) + \mathrm{i}\omega\boldsymbol{M}]Z(x,\omega) + g(x,\omega). \tag{6.17}$$

One of the advantages of studying the problem (6.17) generated by the CGL equation is that the Floquet decomposition can be constructed explicitly. We find that the flow operator $\boldsymbol{\Phi}(x,\omega)$ has the explicit decomposition

$$\boldsymbol{\Phi}(x,\omega) = \boldsymbol{Q}(x)\boldsymbol{H}(x,\omega), \tag{6.18}$$

where $\boldsymbol{Q}(x)$ is as defined in (6.13). The operator $\boldsymbol{H}(x,\omega)$ satisfies

$$\boldsymbol{H}_x = [-\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q}_x + \boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{A}(x) + \mathrm{i}\omega\boldsymbol{M})\boldsymbol{Q}]\boldsymbol{H}, \quad \text{with} \quad \boldsymbol{H}(0,\omega) = \boldsymbol{I}_4. \tag{6.19}$$

A straightforward calculation shows that

$$\left.\begin{aligned} -\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q}_x = \alpha\begin{pmatrix} \boldsymbol{J}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_2 \end{pmatrix}, \quad \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{A}(x)\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{m}^{-1} \\ -\boldsymbol{I}_2 + \rho^2\boldsymbol{n} + 2\boldsymbol{n}\xi\xi^{\mathrm{T}} & c\,\boldsymbol{m}^{-1} \end{pmatrix}, \\ \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{Q} = \boldsymbol{M}. \end{aligned}\right\} \tag{6.20}$$

Consequently, $\boldsymbol{H}(x,\omega)$ satisfies the equation

$$\boldsymbol{H}_x = \boldsymbol{B}(\omega)\boldsymbol{H} \quad \text{with} \quad \boldsymbol{H}(0,\omega) = \boldsymbol{I}_4, \tag{6.21}$$

and

$$\boldsymbol{B}(\omega) = \begin{pmatrix} \alpha \boldsymbol{J}_2 & \boldsymbol{m}^{-1} \\ -(\mathrm{i}\omega + 1)\boldsymbol{I}_2 + \rho^2 \boldsymbol{n} + 2\boldsymbol{n}\xi\xi^{\mathrm{T}} & \alpha \boldsymbol{J}_2 + c\boldsymbol{m}^{-1} \end{pmatrix}. \tag{6.22}$$

Hence,

$$\boldsymbol{H}(x,\omega) = \mathrm{e}^{\boldsymbol{B}(\omega)x}. \tag{6.23}$$

The dispersion relation function (cf. § 4, equations (4.7) and (4.12)) is defined by

$$D(k,\omega) = \det[\mathrm{i}k\boldsymbol{I}_4 - \boldsymbol{B}(\omega)] = \omega^2 - 2\mathrm{i}p(k)\omega - q(k), \tag{6.24}$$

where

$$\left.\begin{aligned} p(k) &= -k^2 - \rho^2 - 2\mathrm{i}k\alpha a - \mathrm{i}kc, \\ q(k) &= (1+a^2)k^2(k^2 - 4\alpha^2) - 2\rho^2(-1+ab)k^2 - c^2k^2 \\ &\qquad -4\alpha ack^2 + 4\mathrm{i}\alpha(a+b)k\rho^2 + 2\mathrm{i}ck(\rho^2 + k^2). \end{aligned}\right\} \tag{6.25}$$

The dispersion relation function (6.24), (6.25) is equal to the dispersion relation function obtained by using a normal mode analysis (cf. Stuart & DiPrima 1978, § 3–5). Moreover, since $\omega$ and $k$ represent parameters in the Fourier–Laplace transform, the time-asymptotics of the solution can also be obtained from the dispersion relation. In fact, we will show that a study of the initial value problem and use of the theory of §§ 2–4 leads to some interesting new results on the long-time behaviour of the instabilities of the spatially periodic travelling waves of the CGL equation that are not obtainable from the normal mode analysis alone.

### (a) *Absolute nature of the Eckhaus instability*

A model for analysing the Eckhaus instability is the Ginzburg–Landau equation with real coefficients (Eckhaus 1965; Stuart & DiPrima 1978)

$$\Psi_t = \Psi + \Psi_{xx} - |\Psi|^2\Psi, \quad \text{with} \quad x \in \mathbb{R} \quad \text{and} \quad t > 0, \tag{6.26}$$

where $\Psi(x,t)$ is complex-valued. The periodic basic state of (6.26) is

$$\Psi(x,t) = \xi \mathrm{e}^{\mathrm{i}\alpha x}, \quad \text{with} \quad \xi \in \mathbb{C}, \quad \alpha \in \mathbb{R} \quad \text{and} \quad |\xi|^2 + \alpha^2 = 1. \tag{6.27}$$

Linearizing (6.26) about the state (6.27) and studying the linear stability problem shows that the state (6.27) is linearly unstable for $\alpha^2 > \frac{1}{3}$ resulting in the Eckhaus boundary $\alpha^2 = \frac{1}{3}$ (Eckhaus 1965). In this section we will study the time asymptotics of the solutions of the equation (6.26) linearized about the state (6.27) and apply the formalism of §§ 2–4 to demonstrate that the Eckhaus instability is an absolute instability. Noting that the real Ginzburg–Landau equation is obtained from the CGL equation by taking $a = b = 0$ we find immediately that the dispersion relation function reads

$$\begin{aligned} D(k,\omega) &= \det[\mathrm{i}k\boldsymbol{I}_4 - \boldsymbol{B}(\omega)|_{a=b=0}] \\ &= \omega^2 + 2\mathrm{i}(k^2 + \rho^2)\omega - k^2(k^2 - 4\alpha^2 + 2\rho^2). \end{aligned} \tag{6.28}$$

The matrix $[\mathrm{i}k\boldsymbol{I}_4 - \boldsymbol{B}(\omega)]^{-1}$ which appears in the inverse Fourier–Laplace transform has the form

$$[\mathrm{i}k\boldsymbol{I}_4 - \boldsymbol{B}(\omega)]^{-1} = \frac{1}{D(k,\omega)}\boldsymbol{B}_1(k,\omega). \tag{6.29}$$

The entries of the matrix $\boldsymbol{B}_1(k,\omega)$ are polynomials in $k$ and $\omega$ of the degree of at most 3 in $k$ and of at most 2 in $\omega$.

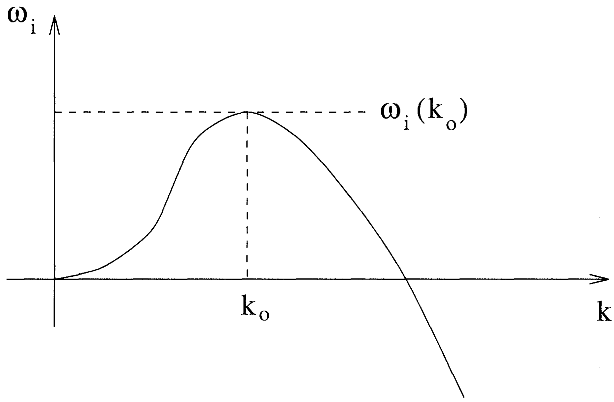


Figure 4. Graph of $\omega_{\rm i}(k)$ for real $k$ for the Eckhaus instability with $\omega_{\rm i}(k_0) = (\gamma/2\alpha)^2$.

The dispersion relation $D(k,\omega) = 0$ can be completely analysed. The frequency as a function of the wave number is given by

$$\omega = -{\rm i}(k^2+\rho^2) \pm {\rm i}\sqrt{\rho^4 + 2k^2(\rho^2+\gamma)}, \tag{6.30}$$

where $\gamma = 3\alpha^2 - 1$, and the expression for the wave number as a function of the frequency is

$$k = \pm\sqrt{{\rm i}\omega + 2\alpha^2 - \rho^2 \pm \sqrt{4{\rm i}\omega\alpha^2 + (\rho^2-2\alpha^2)^2}}. \tag{6.31}$$

The minus sign in (6.30) corresponds to a mode which is always stable. The potentially unstable mode is

$$\omega = -{\rm i}(k^2+\rho^2) + {\rm i}\sqrt{\rho^4 + 2k^2(\rho^2+\gamma)}. \tag{6.32}$$

If $\gamma < 0$ then ${\rm Im}\,\omega < 0$ for all real $k$ resulting in stability. The curve $\gamma = 0$ is the Eckhaus boundary. For $\gamma > 0$ the graph of ${\rm Im}\,\omega(k) = \omega_{\rm i}(k)$ for positive real $k$ is shown in figure 4.

The function $\omega_{\rm i}(k)$ attains its maximum of

$$\omega_{\rm im} = \left(\frac{\gamma}{2\alpha}\right)^2 \quad \text{at} \quad k_0^2 = \frac{2\rho^2\gamma + \gamma^2}{2(\rho^2+\gamma)}.$$

Therefore, the imaginary parts of all unstable frequencies are bounded from above. Since the set of all unstable wave numbers is a finite interval, the real parts of all unstable frequencies are bounded from below and above as well. It can be easily seen that the wave numbers $k_j(\omega)$, $1 \leqslant j \leqslant 4$, given by (6.31) are uniformly bounded away from the real $k$-axis for $\omega$ with ${\rm Im}\,\omega > \omega_{\rm im}$.

In order for the theory of §§2–4 to be applicable to this flow it is necessary to assume that the initial condition (2.5) is zero, i.e. $Z_0(x) = 0$. Then the norm of $T(k,\omega)$ in the corresponding inverse Fourier and Laplace transforms decays faster than $|\omega|^{-n}$, for any $n > 0$, when $\omega$ tends to infinity in both directions in the strip $\mathcal{S}_\sigma$, for any $\sigma > 0$. Consequently, the norm of $v(x,\omega)$ given in (3.15) decays faster than $|\omega|^{-n}$, for any $n > 0$, when ${\rm Re}\,\omega \to \pm\infty$, with ${\rm Im}\,\omega = \text{const.} > \omega_{\rm im}$. Since $\boldsymbol{Q}(x,\omega)$ in (3.16) in this example is independent of $\omega$, the integral (3.16) converges. If $Z_0(x) \neq 0$ we cannot deduce that $\|v(x,\omega)\|$ decays in general, when ${\rm Re}\,\omega \to \pm\infty$, with ${\rm Im}\,\omega = \text{const.} > \omega_{\rm im}$. This is because $T(k,\omega)$ in such a case will have a term independent of $\omega$ coming from a non-zero $Z_0(x)$ and some entries of the matrix

$[\mathrm{i}k\boldsymbol{I}_N - \boldsymbol{B}(\omega)]^{-1}$ are ratios of quadratic polynomials in $\omega$. The assumption $Z_0(x) = 0$ does not restrict the scope of the physical applications due to the fact that the source term $G(x,t)$ is allowed to be an arbitrary function with compact support. Since an emergence of any disturbance in a physically realistic situation is not really sudden but has an evolutionary character, every initial-value problem can be reformulated as a problem with a zero initial condition by making a shift in time to a stage at which no disturbance was present. In such a formulation the influence of a non-zero $Z_0(x)$ will be incorporated through the appropriately modified source term $G(x,t)$.

Let $\omega_{rm}$ denote the maximum of $|\operatorname{Re}\omega|$ for all unstable frequencies $\omega$. For real $\omega_0$ with $|\operatorname{Re}\omega_0| > \omega_{rm}$ no crossing of the numbers $k_j(\omega)$, $1 \leqslant j \leqslant 4$, from one to the other side of the real $k$-axis can take place as $\omega \searrow \omega_0$. Therefore, the analytic continuation of the function $R(x_0,\omega)$ in (4.3) from above the Bromwich contour down to the real $\omega$-axis for such $\omega_0$ is given by the same integral (4.3), and the function $\|R(x_0,\omega)\|$ decays faster than $|\omega|^{-n}$, for any $n > 0$, when $\omega$ tends to infinity in both directions in the strip $\mathcal{S}_\sigma$. The latter is due to the decay of the function $\|T(k,\omega)\|$ and the uniform boundedness of the norm $\|[\mathrm{i}k\boldsymbol{I}_N - \boldsymbol{B}(\omega)]^{-1}\|$ for such $\omega$ and all real $k$. Consequently, the theory of §§ 2–4 leading to the integral (4.5), the collision criterion, and the dominant term of the asymptotics (4.14) are applicable in this example.

If a point $(k_0, \omega_0)$ is to contribute to an absolute instability the necessary condition (4.9) has to be satisfied. The equation

$$\frac{\partial}{\partial k} D(k,\omega) = 0$$

for $D(k,\omega)$ given by (6.28) reads

$$D_k(k,\omega) = 4k(\mathrm{i}\omega - k^2 + \gamma) = 0. \tag{6.33}$$

Its solutions are $k = 0$ and

$$\omega = \mathrm{i}(\gamma - k^2). \tag{6.34}$$

But $\operatorname{Im}\omega \leqslant 0$ when $k = 0$, and the equation $D(k,\omega) = 0$ for the point (6.34) gives

$$k_0^2 = \gamma - \left(\frac{\gamma}{2\alpha}\right)^2 \quad \text{and} \quad \omega_0 = \mathrm{i}\left(\frac{\gamma}{2\alpha}\right)^2 \tag{6.35}$$

showing that $\omega_0$ is in the upper half $\omega$-plane. It remains to verify the pinching condition. The maximum value of $\operatorname{Im}\omega$ for $k \in \mathbb{R}$ is exactly the value $\operatorname{Im}\omega_0$. Therefore, when $\omega \searrow \omega_0$, the colliding roots cannot cross the real $k$-axis for $\omega$ above $\omega_0$. A short calculation using $\omega = \omega_0 + \mathrm{i}y$, with positive $y$ shows that

$$k - k_0 = \pm\mathrm{i}\frac{\alpha}{k_0}\sqrt{y} + o(\sqrt{y}) \quad \text{as} \quad y \to 0_+. \tag{6.36}$$

Hence, the roots that collide at the point $k_0 \in \mathbb{R}$ originate on opposite sides of the real $k$-axis. This verifies that the Eckhaus instability is an absolute instability for any $\alpha$ in the unstable region $\frac{1}{3} < \alpha^2 < 1$.

### (b) *Absolutely and convectively unstable periodic solutions of the complex Ginzburg–Landau equation*

When the coefficients in the Ginzburg–Landau equation are complex $(a^2 + b^2 \neq 0)$ new features in the linear stability analysis and in the evolution of wave packets in the spatially periodic flow appear. A treatment of the normal mode stability of periodic solutions of the Ginzburg–Landau equation with complex coefficients is given in Stuart & DiPrima (1978). In the normal mode analysis, the main new feature is that the region of linearly stable periodic states is not always equal to the Eckhaus range ($\alpha^2 < \frac{1}{3}$) but can be either wider or narrower and can even vanish. However, a consideration of the initial-value problem, and an analysis of the unstable wave packets, using the theory of §§ 2–5, show that the unstable periodic waves can be either convectively or absolutely unstable. Moreover, the transition between the two types of instability is not predictable from the normal-mode linear stability analysis alone.

When $a^2 + b^2 \neq 0$, the basic state of the complex Ginzburg–Landau equation is stationary and periodic in a frame of reference moving at speed $-c$ relative to the absolute frame of reference, where $c$ is determined by the position along the branch (cf. equation (6.5)). Therefore, there are four frames of reference that will arise naturally in the analysis: ($a$) the absolute frame of reference, ($b$) the frame of reference in which the basic state is stationary, ($c$) the frame of reference moving with the group velocity of the wave packet relative to the absolute frame of reference, and ($d$) the frame of reference of the observer.

In the frame of reference in which the basic flow is stationary, i.e. in that moving at the speed $-c$ relative to the absolute frame of reference, the dispersion relation function is given by (6.24), (6.25). By introducing an observer moving at the speed $V \in \mathbb{R}$ relative to this moving frame of reference the dispersion relation is modified to

$$D(k, \omega + kV) = (\omega + kV)^2 - 2\mathrm{i}p(k)(\omega + kV) - q(k). \tag{6.37}$$

When looking for an absolute instability the relevant frame of reference is the absolute frame of reference which is obtained by taking $V = -c$. When $V = -c$, the dispersion relation function $D(k, \omega - kc)$ is given by the expressions (6.24), (6.25) with $c$ set to zero. Therefore, the applicability of the theory in §§ 2–4 to the CGL equation can be justified under the same assumptions and in the same manner as for the real GL equation.

In the CGL equation we find that both absolutely and convectively unstable wave packets appear in the spatially periodic flow. We will not give a complete account for the parameter space but select a representative point, in particular $a = 1$ and $b = -3$. At the point $a = 1$ and $b = -3$ there is a branch of finite-amplitude travelling waves as shown in figure 5.

As $\ell$ varies along the branch (6.6) from 0 to $\frac{1}{2}\pi$ the wavespeed $c(\ell)$ of the branch of travelling waves varies monotonically from $-1$ to $-\infty$, with $c(\ell) \to -\infty$ as $\ell \to \frac{1}{2}\pi$. The theory of §§ 2–4 applied to this branch leads to the following results. As the amplitude $\rho(\ell)$ is increased from the trivial state ($\ell = 0$) we find that initially the basic state is absolutely unstable for

$$0 < \ell < \ell_1 \approx 0.977. \tag{6.38}$$

For larger $\ell$,

$$\ell_1 < \ell < \ell_2 = \arccos\sqrt{\tfrac{1}{6}} \approx 1.152, \tag{6.39}$$

the wave is absolutely stable but convectively unstable and when

$$\ell_2 < \ell < \tfrac{1}{2}\pi, \tag{6.40}$$

the basic state is linearly stable.

Before verifying the picture in figure 5 we illustrate its consequences for flowfields

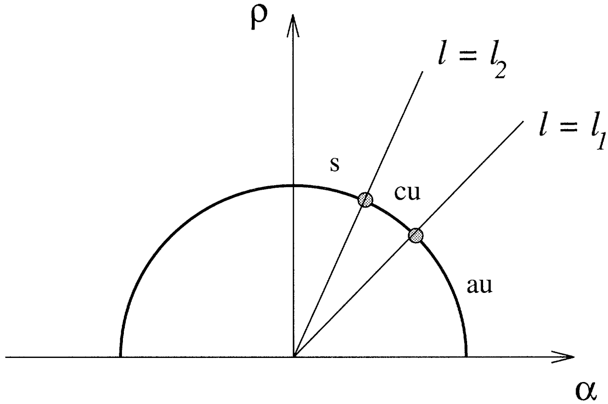


Figure 5. Branch of finite-amplitude travelling waves of CGL for $a = 1$ and $b = -3$. $\ell_1 \approx 0.977$, $\ell_2 \approx 1.152$. Regions: au, absolutely unstable; cu, convectively unstable but absolutely stable; s, stable.

where the CGL equation is a model. The CGL equation is a model equation for near critical hydrodynamic stability problems where a homogeneous state loses stability at the neutral curve as shown in figure 1 in the introduction. The CGL equation models the nonlinear dynamics of the waves near the minimal point on the neutral curve corresponding to the lowest value of $R$, where $R$ is a control parameter such as a Reynolds number. Projecting the results of figure 5 onto figure 1 shows that the unstable region of spatially periodic travelling waves, that exists, is composed of two distinct regions. One of these regions, namely the exterior band, is absolutely unstable, similar to the entire unstable band in the Eckhaus instability, while the interior band is convectively unstable, but absolutely stable. This result is only verified in the present work for $a = 1$ and $b = -3$, but we anticipate it to be valid for an open subset of the $(a, b)$-plane.

It is interesting to note that here, as in the case of the Eckhaus instability, for every wavenumber $\alpha$ different from the critical one, the trivial state, which is stable for a subcritical value of $R$, becomes unstable and gives rise to an absolutely unstable travelling wave, when $R$ increases through its critical value for this $\alpha$. A transition from the stable trivial state to a stable travelling wave takes place only at the critical wavenumber $\alpha$. Because of the nature of the absolute instability, an absolutely unstable state acts as a repeller, in the phase space of the time-evolution, at every location in space; that is, such a state is a physical repeller in time globally in space. We can claim, therefore, that the presence of the absolutely unstable band in figure 1 is a theoretical explanation for the rapid selection of the critical wavelength at slightly supercritical values of $R$ observed in the media that are governed by the Ginzburg–Landau equation (cf. Koschmieder (1974) for the case of Bénard convection). For such $R$, a perturbation of the homogeneous state by a disturbance in which a continuum of wavelengths is present causes a rapid divergence from spatially global repellers, i.e. a rapid destruction of the components of the perturbation that are absolutely unstable, and a rapid convergence to the spatially periodic state corresponding to the critical wavenumber. This scenario is suggested from the theoretical results. In other words, the spatially global repelling nature of an absolutely unstable state may be a triggering mechanism for the rapid wavenumber selection in this case.

The change of stability for $\ell = \ell_2$ is easily verified using the normal mode linear stability theory. In fact, when $a = 1$ and $b = -3$ setting $D(k, \omega) = 0$ results in

$$\operatorname{Im} \omega = -\frac{4}{\rho^2}(1 - 6\alpha^2)k^2 + \mathcal{O}(k^4), \tag{6.41}$$

with the change of stability occurring for small $|k|$ when $\alpha^2 = \cos^2 \ell = \frac{1}{6}$. One can then verify that $\operatorname{Im} \omega < 0$ when $\alpha^2 < \frac{1}{6}$ for all $k$.

To understand the transition from absolute to convective instability at $\ell = \ell_1$ the necessary and sufficient condition for absolute instability is applied. First note that in the limit $\lim_{\ell \to 0} \rho(\ell) = 0$ (the limit as the amplitude of the branch of travelling waves goes to zero) the multiple roots in $k$ of the dispersion relation function can be found exactly. In this limit $D(k, \omega + kV) = 0$ with $V = -c$ (cf. (6.24), (6.25)) results in

$$\lim_{\rho \to 0} \omega(k) = -\mathrm{i}k(k + 2\mathrm{i}) \pm k(k - 2\mathrm{i}). \tag{6.42}$$

We have

$$\lim_{\rho \to 0} \omega'(k) = 2[-\mathrm{i}k + 1 \pm (k - \mathrm{i})], \tag{6.43}$$

and, therefore, in the limit as $\rho \to 0$, there exist two saddle points of the frequency function at $k = \pm 1$ with

$$\omega_0(k)|_{k=\pm 1} = \mathrm{i} \pm 1. \tag{6.44}$$

The maximum of $\operatorname{Im} \omega(k)$ for real $k$ is equal to $\operatorname{Im} \omega_0(\pm 1) = 1$. Let $\omega = \omega_0 + \mathrm{i}y$, with $y$ real and positive. Then $y > 0$ splits the double $k$ root and perturbs the two values of $k$ into the upper and lower half of the complex $k$-plane. Therefore, both saddle points in the $k$-plane for $\omega = \omega_0$ give rise to an absolute instability.

We now evaluate the movement of the values of $\omega(k)$ at the saddle points found above in the $\omega$-plane along the branch of travelling waves. We refer to these values as saddle points in the $\omega$-plane. The double roots in $k$ of the dispersion relation function $D(k, \omega + kV)$ satisfy

$$\left.\begin{aligned} D(k, \omega + kV) = 0, \\ \frac{\mathrm{d}}{\mathrm{d}k} D(k, \omega + kV) \equiv D_k(k, \omega + kV) + V D_\omega(k, \omega + kV) = 0. \end{aligned}\right\} \tag{6.45}$$

For $\rho > 0$ and $D(k, \omega)$ given by (6.24), (6.25) equations (6.45) combine to give a sixth order, in $k$, equation at each value of $\ell$ and $V$. Therefore, these roots cannot in general be obtained analytically. In figure 6 the movement of the two saddle points originating at $\omega = \pm 1 + \mathrm{i}$ is followed numerically in the complex $\omega$-plane, as $\ell$ varies along the branch of travelling waves, for the case $V = -c$. Beginning at $\omega = \pm 1 + \mathrm{i}$, the saddle points move, as $\ell$ is increased from $\ell = 0$, symmetrically while their imaginary part decreases monotonically from its maximum value of 1. At the critical value of $\ell$, denoted by $\ell_1$, the saddle points cross the real axis in their movement from the upper to the lower half plane in the complex $\omega$-plane. The sign of the imaginary part of the two saddle points in the $\omega$-plane changes at $\ell = \ell_1$ from plus to minus.

It is verified computationally that the two saddle points satisfy the pinching condition for all $\ell \in (0, \ell_1)$ (see also figures 7 and 8) and, therefore, give rise to the absolute instability in this range. Moreover, we have verified numerically that at all the other saddle points for all $\ell \in (0, \ell_1)$ either $\omega$ has a non-positive imaginary part or the collision criterion is not satisfied. Therefore, for all $\ell \in (0, \ell_1)$ the two saddle





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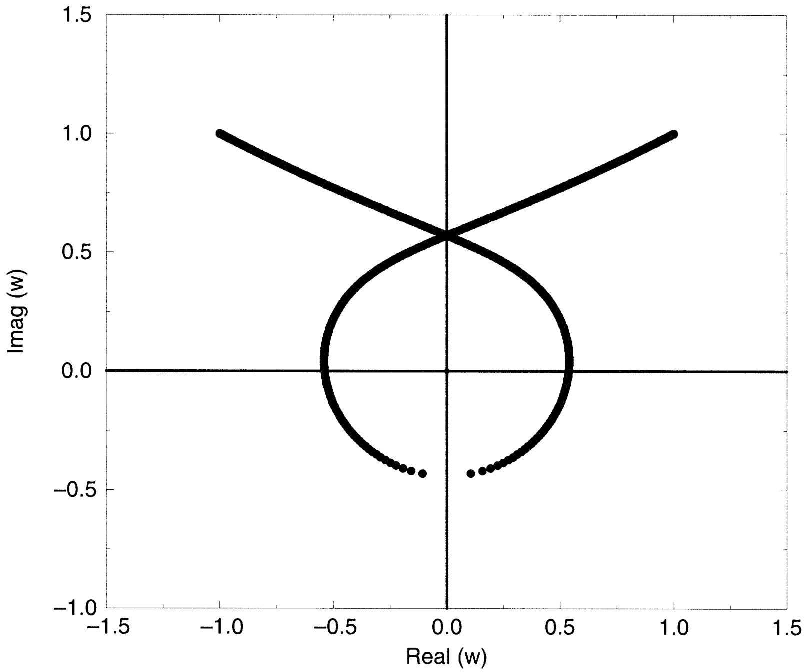


Figure 6. Movements of the two saddle points, that make the dominant contribution to the asymptotics, in the complex $\omega$-plane from the upper to the lower half plane as $\ell$ increases from 0 to $\frac{1}{2}\pi$ for CGL when $a = 1$ and $b = -3$.

points shown in figure 6 make the dominant contribution to the asymptotics of the solution. When $\ell = \ell_1$ there is a stabilization of the two saddle points. It is important to note that, although the saddle points stabilize at $\ell = \ell_1$, the basic state is still unstable in the region $\ell_1 < \ell < \ell_2$. In this region the periodic wave is *absolutely stable* but *convectively unstable*. Note, however, that no qualitative change of the normal modes takes place at $\ell = \ell_1$. Thus, the method of normal modes produces no information with regard to the absolute or convective nature of the instability. Further decrease of $\ell$ through its critical value $\ell_2$ causes stabilization of the normal modes, and, therefore, of the periodic travelling wave.

To illustrate the verification of the pinching condition, in figures 7 and 8 the images of the lines $\{\omega \mid \operatorname{Im}\omega = \sigma\}$ in the $k$-plane for $k = k_j(\omega)$, $1 \leqslant j \leqslant 4$, where $k_j(\omega)$, $1 \leqslant j \leqslant 4$, denote the $k$-roots of the dispersion relation function $D(k, \omega)$, are shown in the complex $k$-plane for different values of $\operatorname{Im}\omega = \sigma$, for two representative values of $\ell$.

Figure 7 shows a typical set of diagrams for the region $0 < \ell < \ell_1$. Here $\ell = 0.5$ and is in the absolutely unstable region of the branch. In figure 7*a* the images of the Bromwich contour are shown for $\sigma = 1.5$, which is greater than the maximum of $\operatorname{Im}\omega(k)$ for real $k$. We clearly identify the branches that lie on different sides of the real $k$-axis. In figure 7*b* the imaginary part of $\omega$ is reduced to the value at the unstable saddle point $\operatorname{Im}\omega_0 = 0.6390$. By comparing figures 7*a* and 7*b* one can see that, as $\operatorname{Im}\omega$ decreases from 1.5 down to $\operatorname{Im}\omega_0$, two images of the Bromwich contour coming from above the real $k$-axis collide with two images that originate from below the axis. Therefore, the pinching condition is satisfied for these unstable saddle points, and the flow is unstable. The same situation was observed for the entire range $0 < \ell < \ell_1$.

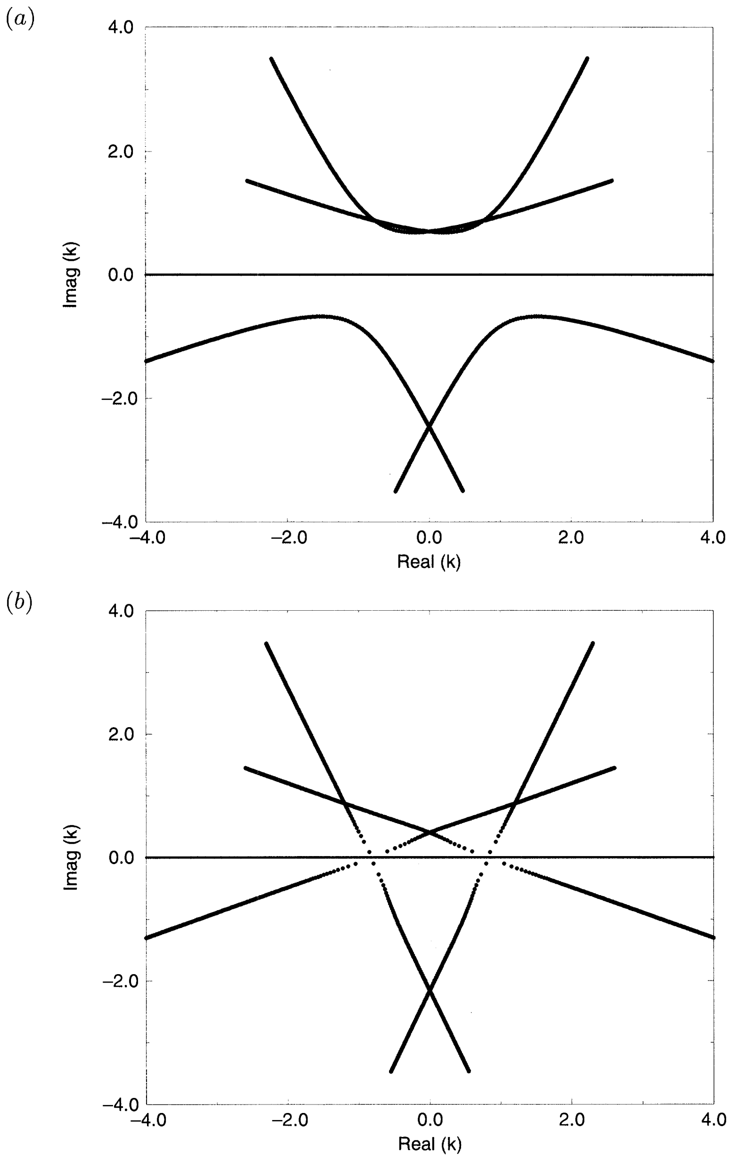


Figure 7. Images of the lines of constant $\mathrm{Im}\,\omega$ in the complex $k$-plane for periodic states of CGL in the au-region when $\ell = 0.5$. (*a*) $\mathrm{Im}\,\omega = 1.5$ (the Bromwich contour); (*b*) $\mathrm{Im}\,\omega = 0.6390$.

When $\ell_1 < \ell < \ell_2$ the basic state is convectively unstable. To illustrate this we present in figure 8 the images of the Bromwich contours for a representative value of $\ell$ in this range: $\ell = 1$. In figure 8*a* the images of the Bromwich contour in the complex $k$-plane are shown for $\mathrm{Im}\,\omega = 1.0$, which is greater than the maximum of $\mathrm{Im}\,\omega(k)$ for real $k$. Again, as in figure 7*a*, the branches that lie on different sides of the real $k$-axis are clearly identified. In figure 8*b* the images for $\mathrm{Im}\,\omega = 0$, i.e. the images of the real $\omega$ axis, in the $k$-plane are shown. It is clearly seen that two images coming from above the real $k$-axis collided as $\mathrm{Im}\,\omega$ decreased from 1.0 down to zero. Two images originating from below the real $k$-axis experienced a certain change and moved. In

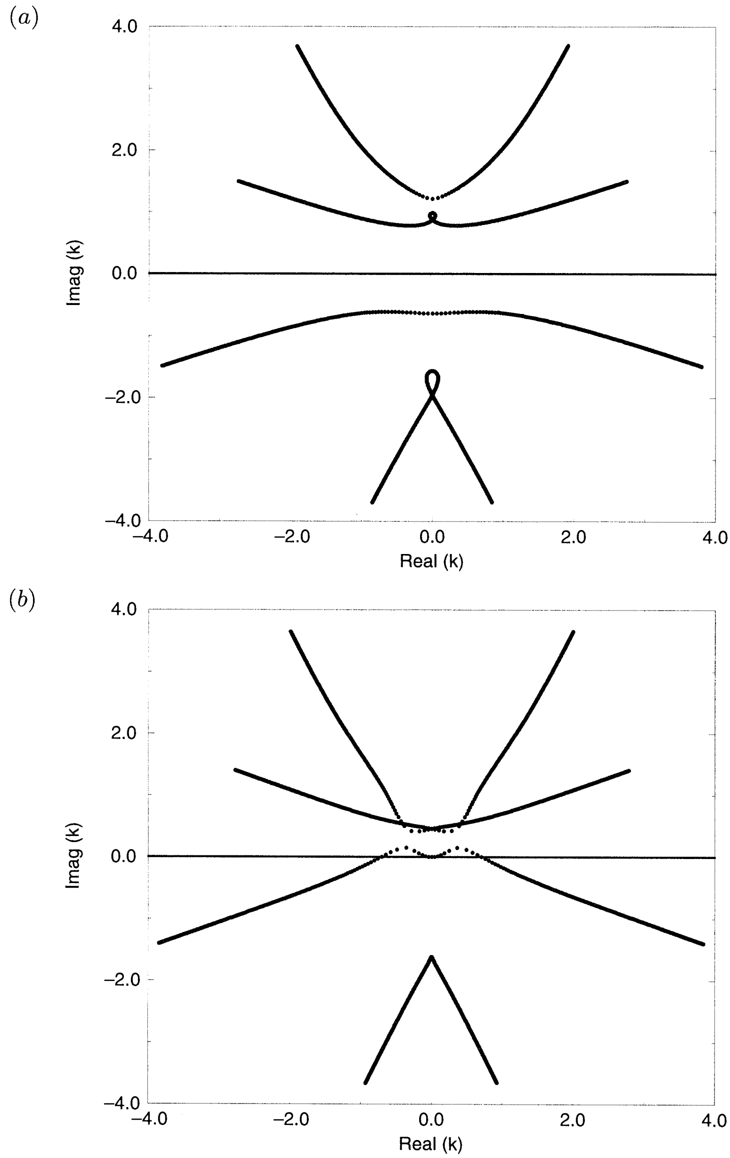


Figure 8. Images of the lines of constant $\mathrm{Im}\,\omega$ in the complex $k$-plane in the cu-region for $\ell = 1$. ($a$) $\mathrm{Im}\,\omega = 1$ (the Bromwich contour); ($b$) $\mathrm{Im}\,\omega = 0$.

particular, the upper one of them crossed the real $k$-axis from below. However, no collision of the images coming from opposite sides of the real $k$-axis took place. The flow is, therefore, absolutely stable.

It is important to emphasize that, although a collision of the images for a marginally positive $\mathrm{Im}\,\omega$ occurs, this collision does not cause absolute instability, since the colliding images originate on the same side of the real $k$-axis in the limit $\mathrm{Im}(\omega - \omega_0) \to +\infty$, for $\mathrm{Re}(\omega - \omega_0) = 0$. The saddle points of $\omega = \omega(k)$ which result from this collision have a marginally positive imaginary part, but they make no contribution to absolute instability, because they do not satisfy the pinching condition. Thus, there are no

(*c*)

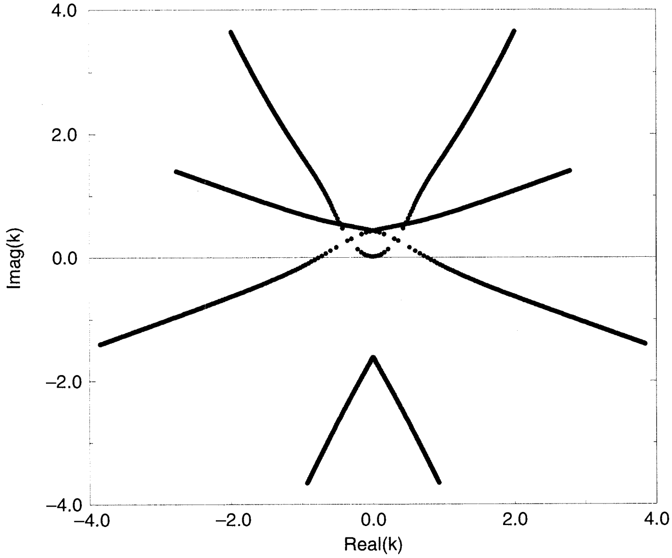


Figure 8. (*c*) $\mathrm{Im}\,\omega = -0.0427$.

singularities of the corresponding function $R(x_0, \omega)$ (4.3) in the infinite strip $\omega \in \mathcal{S}_\sigma$, although the wave is unstable.

Lowering the value of $\omega_\mathrm{i}$ further (in particular till $\omega_\mathrm{i} = -0.0427$ in this case) one sees that the collision of the images originating on opposite sides of the real $k$-axis eventually takes place but it corresponds to a stable frequency. Therefore, the dominant contribution to the asymptotics of the solution along the ray $x = x_0$ is decaying in time. Analogous results were obtained for all $\ell$ in the range $\ell_1 < \ell < \ell_2$. Since the normal modes in this range are unstable, the flow is convectively unstable for $\ell \in (\ell_1,\ \ell_2)$. We remark that algebraically growing wave packets are not found for the CGL equation at the neutral point $\ell = \ell_2$. A treatment of the signalling problem for the CGL equation is left for future work.

Finally, we remark on some numerical computations of Aranson *et al.* (1992) of the saddle points of the dispersion relation of the CGL equation and their connection with the spatiotemporal chaos observed in dynamic simulations of CGL. Without addressing the initial-value problem, Aranson *et al.* (1992) computed saddle points for the dispersion relation of the CGL equation and conjectured that an unstable saddle point corresponds to an absolutely unstable travelling wave solution. They then related the regions in the parameter space, where a particular saddle point is unstable, to the complexity of the nonlinear spatiotemporal behaviour observed in dynamic simulations. The condition for absolute instability used by Aranson *et al.* is that there exists an unstable saddle point $k_0$ of $\omega = \omega(k)$, that is, $\mathrm{d}\omega(k_0)/\mathrm{d}k = 0$, with $\mathrm{Im}\,\omega(k_0) > 0$, that satisfies $\mathrm{Im}(\mathrm{d}^2\omega(k_0)/\mathrm{d}k^2) < 0$. This condition is in general incorrect; it is neither sufficient nor necessary for an absolute instability. In particular, a saddle point satisfying this condition does not necessarily satisfy the correct collision criterion (cf. § 4) and, as shown in the present work, a saddle point that satisfies the collision criterion does not necessarily satisfy the above condition. On the other hand, as shown in § 6 *a*, there is only a single unstable saddle point corresponding to the Eckhaus instability and, moreover, it satisfies the collision criterion derived in § 2–5, so the instability is absolute. Fortuitously, this particular saddle point also satisfies the condition used by Aranson *et al.* (1992). However, it is not assured that all unstable saddle points are of this type for all regions of the parameter space for the CGL equation. In any case the theory of § 2–5 gives precise conditions for absolute and convective instability in the spatially periodic case and should be practically verifiable for all values of parameters in the CGL equation.

## 7. Application to spatially periodic states of the Navier–Stokes equations

A classification of the instabilities of spatially periodic travelling waves of the Navier–Stokes equations is of fundamental interest for studying the long-time asymptotics of secondary instabilities in shear flows and their role in the transition to turbulence. The main theoretical difficulty with extending the theory of §§ 2–5 to the Navier–Stokes equations is that the presence of a cross section leads to derivatives in the operator $F(Z)$. In this section, the cross section is discretized and a numerical procedure, based on the theory of §§ 2–5, for classifying absolute and convective instabilities is developed.

Consider a fluid domain in the $(x, y)$-plane, with $-\infty < x < \infty$ and $y \in (h_1, h_2)$, where $h_2 - h_1 > 0$, and for simplicity $h_1$ and $h_2$ are finite. The Navier–Stokes equations in dimensionless form in this strip can be written as

$$\left.\begin{aligned} u_x + v_y &= 0, \\ u_t + uu_x + vu_y + p_x - \frac{1}{R}(u_{xx} + u_{yy}) &= 0, \\ v_t + uv_x + vv_y + p_y - \frac{1}{R}(v_{xx} + v_{yy}) &= 0, \end{aligned}\right\} \tag{7.1}$$

with suitable boundary conditions for $(u, v)$ along the edge of the strip. The variables $u$ and $v$ represent the components of the fluid velocity in the $x$- and $y$-directions respectively, $p$ is the pressure, and $R$ is the Reynolds number. By introducing a new variable $\mathcal{V} = v_x$ the system (7.1) can be rewritten

$$\left.\begin{aligned} u_x &= -v_y, \\ v_x &= \mathcal{V}, \\ u_t + p_x &= uv_y - vu_y + \frac{1}{R}(-\mathcal{V}_y + u_{yy}), \\ -Rv_t + \mathcal{V}_x &= -v_{yy} + Rp_y + R(vv_y + u\mathcal{V}), \end{aligned}\right\} \tag{7.2}$$

which is put in the form

$$\boldsymbol{M}Z_t + Z_x = F(Z) \tag{7.3}$$

by setting

$$\boldsymbol{M} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -R & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} u \\ v \\ p \\ \mathcal{V} \end{pmatrix} \tag{7.4}$$

and

$$F(Z) = \begin{pmatrix} -v_y \\ \mathcal{V} \\ \dfrac{1}{R}(-\mathcal{V}_y + u_{yy}) + uv_y - vu_y \\ Rp_y - v_{yy} + R(vv_y + u\mathcal{V}) \end{pmatrix}. \tag{7.5}$$

Note that for the Navier–Stokes equations the operator $\boldsymbol{M}$, though linear, acts on a functional space associated with the $y$-direction where the basic flow has shear. Even though $F(Z)$ involves derivatives, the derivatives are with respect to the vertical (finite) direction and so abstractly the form (7.3) is equivalent to (2.1). Another way to see this is to introduce a finite-dimensional discretization of the channel cross section (cf. below). In the representation (7.3) the simplest class of uniform (or parallel) flows satisfies $F(Z) = 0$ along with the boundary conditions at $y = h_1$ and $y = h_2$.

Suppose that, in addition to a basic parallel flow, there exists a finite-amplitude spatially periodic travelling wave, with the sum of the two solutions denoted by $W(x, y)$, that satisfies

$$(\boldsymbol{I} - c\boldsymbol{M})W_x = F(W), \tag{7.6}$$

where $\boldsymbol{I}$ is the identity operator. Such a state can arise as a result of a bifurcation from a parallel flow, e.g. plane channel flow with various boundary conditions (moving wall or compliant wall for example).

Linearization of

$$\boldsymbol{M}Z_t + (\boldsymbol{I} - c\boldsymbol{M})Z_x = F(Z). \tag{7.7}$$

about the state $W(x, y)$ gives

$$\boldsymbol{M}Z_t + (\boldsymbol{I} - c\boldsymbol{M})Z_x = \boldsymbol{A}(x, y)Z, \quad \text{with} \quad \boldsymbol{A}(x, y) = DF(W). \tag{7.8}$$

Here $Z = Z(x, y, t)$. Formal application of the Laplace transform as in § 2 to (7.8) results in

$$Z_x(x, y, \omega) = (\boldsymbol{I} - c\boldsymbol{M})^{-1}[\boldsymbol{A}(x, y) + \mathrm{i}\omega\boldsymbol{M}]Z(x, y, \omega) + g(x, y, \omega), \tag{7.9}$$

where $g(x, y, \omega) = \boldsymbol{M}Z|_{t=0}$ plus a source term if present.

The equation (7.9) is a generalization of the equation (2.8) to the case of a periodic flow with shear. However, the essential feature of (7.9) compared to (2.8) is that, for a fixed $x$, $\boldsymbol{A}(x, y)$ in the former is an elliptic differential operator in $y$ rather than a matrix operator $\boldsymbol{A}(x)$ in the latter. Although the theory can be generalized to this case, the Floquet theory for elliptic operators requires different methods (cf. Kuchment 1993). The main new feature is that, since $\boldsymbol{I}\partial_x - (\boldsymbol{I} - c\boldsymbol{M})^{-1}[\boldsymbol{A}(x, y) + \mathrm{i}\omega\boldsymbol{M}]$ is an elliptic operator, the homogeneous part of the equation in (7.9) is ill-posed as an evolution equation and, therefore, the concept of a flow operator and a monodromy operator do not in general exist. However, although the Floquet transform does not generalize, the Floquet decomposition goes through under suitable hypotheses.

In order to apply the theory of §§ 2–5, the difficulties associated with the elliptic operator can be avoided by discretizing the system (7.9) in the $y$-direction. Such a discretization would be necessary in practice anyway since finite-amplitude travelling waves of the Navier–Stokes equations are in general only known numerically. As an





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example of such a discretization let

$$Z(x,y,\omega)=\sum_{n=0}^{N-1}\Upsilon_n(x,\omega)T_n(Y)\ ,\qquad Y=\frac{2y-h_1-h_2}{h_2-h_1}, \tag{7.10}$$

where $\{T_n(Y)\}$ are the Chebyshev polynomials on the interval $-1\leqslant Y\leqslant 1$. We introduce the vector

$$\Upsilon=\begin{pmatrix}\Upsilon_0\\ \Upsilon_1\\ \vdots\\ \Upsilon_{N-1}\end{pmatrix}\in\mathbb{C}^{4N}. \tag{7.11}$$

Then a projection of (7.9), with the corresponding boundary conditions at $y=h_1$ and $y=h_2$, onto the $4N$-dimensional space with the Chebyshev basis, results in the equation

$$\Upsilon_x=\boldsymbol{E}(x,\omega)\Upsilon+g_N, \tag{7.12}$$

where $\boldsymbol{E}(x,\omega)$ is the projection of the operator $(\boldsymbol{I}-c\boldsymbol{M})^{-1}[\boldsymbol{A}(x,y)+\mathrm{i}\omega\boldsymbol{M}]$ and $g_N$ is the projection of the forcing term $g(x,y,\omega)$ in (7.9). The matrix operator $\boldsymbol{E}(x,\omega)$ is periodic in $x$, so the equation (7.12) is now in the form to which the theory of §§ 2–5 is applicable.

It is important to emphasize that although the theory of §§ 2–5 does not provide a direct analytical treatment of the continuous equation (7.9), its discretized version (7.12) can be treated by applying the developed theory because it is a direct analogy of the formulation in §§ 2–5. Thus periodic travelling wave solutions of the Navier–Stokes equations can be investigated for absolute and convective instabilities by combining the theory of §§ 2–5 with numerical computations. The necessity of a discretization in the case of periodic travelling waves in shear flows is quite expected, since even for parallel flows the study of normal mode instabilities of most flows with shear requires numerical methods (cf. Drazin & Reid 1981).

After numerical discretization, an investigation of the absolute and convective instabilities and a treatment of the signalling problem, for periodic solutions of the Navier–Stokes equations, is based on analysis of the movement of the Floquet exponents or multipliers. While the monodromy operator is not well defined in the continuous elliptic case, it is well-defined for the discretized finite-dimensional system (7.12). In principle there are two complementary approaches for a numerical treatment of (7.12). One approach is to compute the monodromy operator of the system (7.12). It should be noted that the initial-value problem for (7.12) will be stiff with the stiffness dependent on the number of discretization points in the $y$-direction. The analysis of absolute and convective instabilities can in this case be carried out under the assumption that the discretization cuts off the eigenvalues $\mathrm{i}k$ of the elliptic operator with increasingly large real part associated with its ill-posedness. The corresponding complex numbers $k$ (with large imaginary part) do not in any case cross the real $k$-axis as $\omega\searrow\omega_0$. On the other hand the eigenvalues $\mathrm{i}k$ with bounded $|\operatorname{Re}\mathrm{i}k|$, that can cross the axis are assumed for physical reasons to be well approximated by the discretization, when it is fine enough.

Let $\boldsymbol{\Phi}(L,\omega)$ be the monodromy matrix of the discrete problem (7.12), where $L$ is the period of $\boldsymbol{E}(x,\omega)$. For a given complex $\omega$ this matrix is computed approximately by integrating numerically the initial-value problem,

$$\boldsymbol{\Phi}_x(x,\omega) = \boldsymbol{E}(x,\omega)\boldsymbol{\Phi}(x,\omega), \qquad \boldsymbol{\Phi}(0,\omega) = \boldsymbol{I}_{4N}, \tag{7.13}$$

on the interval $[0, L]$. All the roots $\mu \in \mathbb{C}$ of

$$\det\left[\boldsymbol{\Phi}(L,\omega) - \mu\,\boldsymbol{I}_{4N}\right] = 0, \tag{7.14}$$

that is, the Floquet *multipliers* are then found by using standard global eigenvalue solvers. Therefore, the pinching condition for absolute and convective instabilities and the causality condition for spatially amplifying waves can be numerically implemented in terms of the Floquet multipliers $\mu$. The contributing wavenumber is then computed as $k_0 = -(\mathrm{i}/L)\ln\mu_0(\omega_0)$, where $\omega_0$ is the contributing frequency and $\mu_0(\omega_0)$ is the contributing Floquet multiplier, which is the collision point in the complex $\mu$-plane satisfying the corresponding pinching condition for the analysis of absolute and convective instabilities (see the discussion following the equation (4.12)), and the Floquet multiplier that crosses the unit circle for the signalling problem with a stationary forcing (see the second paragraph following the equation (5.7)). In this approach convergence of a numerical scheme for integrating the initial value problem (7.13), for any fixed $N$, can be attained by suitably refining the step of the scheme $h(N)$. However, numerical integration of this method sould require a technique such as orthogonalization for eliminating the stiff modes that appear for large $N$.

In the above approach the contributing frequency $\omega_0$ is computed unambiguously but the contributing wavenumber $k_0$ is given up to an additive term $2\pi n/L$, where $n$ is an integer. This is due to the non-uniqueness of the logarithm. However, this ambiguity of $k_0$ does not influence the time asymptotics (4.14) of unstable wave packets. It affects solely the spatial response (5.8) in the signalling problem with a stationary forcing through an arbitrary oscillatory harmonic factor of the form $\mathrm{e}^{2\pi n\mathrm{i}x/L}$, which is bounded and, therefore, of little significance for predicting qualitative properties of turbulent transition. A similar factor will be present in (4.14) introducing an indeterminacy into the local wavelength only. Assuming that the local wavelength $2\pi/\operatorname{Re}k_0$ is a continuous function of the ray velocity $V$, its relative change across the wave packet can still be computed in the analysis.

In the second approach the Floquet *exponents* are computed numerically. Let $\Upsilon(x,\omega) = \mathrm{e}^{\mathrm{i}kx}\widehat{\Upsilon}(x,\omega)$, where $\widehat{\Upsilon}(x,\omega)$ is an $L$-periodic function in $x$ satisfying

$$\widehat{\Upsilon}_x = [\boldsymbol{E}(x,\omega) - \mathrm{i}k\boldsymbol{I}_{4N}]\widehat{\Upsilon}. \tag{7.15}$$

Discretizing (7.15) by using a Fourier series approximation for $\widehat{\Upsilon}$ leads to a standard matrix generalized eigenvalue problem for the Floquet exponent $k$, considered as a function of $\omega$ and other problem-dependent parameters. This second numerical procedure should lead to a convergent algorithm, for $N \to \infty$, provided the length of the truncated Fourier series for $\widehat{\Upsilon}$ increases suitably as a function of $N$.

In the framework of the present approach the discretized problems (7.13), (7.14) can be viewed as a certain analogy, for a spatially periodic basic state, of the discretized Orr–Sommerfeld problem in the homogeneous case. A complete analogy would be possible if the corresponding Floquet operator $\boldsymbol{Q}(x,\omega)$ for the flow field were known. Then the substitution of the form (4.10) into the discretized equation in (7.8) would lead to an equation of the form (4.11), which is a genuine analogy of the discretized Orr–Sommerfeld equation.

## 8. Concluding remarks

The study of the time and space asymptotics of wave packets in spatially periodic media can be applied to many problems and generalized in a number of interesting directions. As a matter of fact, the normal mode or spectral stability analysis of periodic waves has been widely used whereas the wave packet approach would lead to further information. For example, the Fitzhugh–Nagumo equations (cf. Maginu 1978) that model nerve conduction, relative to a frame of reference moving at speed $c$, take the form

$$u_t + cu_x = u_{xx} - f(u) - w, \qquad w_t + cw_x = bu, \tag{8.1}$$

where $f(u) = u(u-a)(u-1)$ and $a$ and $b$ are positive constants. With $v = u_x$ the system (8.1) can be written as a first order system in $x$- and $t$-derivatives of the form $\boldsymbol{M}Z_t + Z_x = F(Z)$ by rewriting (8.1) as

$$u_x = v, \qquad -u_t + v_x = cv + f(u) + w, \qquad \frac{1}{c}w_t + w_x = \frac{b}{c}u, \tag{8.2}$$

and taking

$$\boldsymbol{M} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1/c \end{pmatrix}, \quad Z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathbb{R}^3 \quad \text{and} \quad F(Z) = \begin{pmatrix} v \\ cv + f(u) + w \\ (b/c)u \end{pmatrix}. \tag{8.3}$$

Another example is the laser equations (cf. Swinton & Elgin 1990). Relative to a frame moving at speed $c$ the laser equations take the form

$$u_t + u_x - \frac{1}{c}u_x = \sigma(v-u), \quad v_t + v_x = (r-w)u - v, \quad w_t + w_x = uv - bw, \tag{8.4}$$

where $\sigma$, $r$ and $b$ are given real numbers. The system (8.4) can be written in the form $\boldsymbol{M}Z_t + Z_x = F(Z)$ by taking

$$\boldsymbol{M} = \begin{pmatrix} \nu & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathbb{R}^3 \quad \text{and} \quad F(Z) = \begin{pmatrix} \widehat{\sigma}(v-u) \\ (r-w)u - v \\ uv - bw \end{pmatrix}, \tag{8.5}$$

with $\nu = c/(c-1)$ and $\widehat{\sigma} = \sigma\nu$. For both of the above examples the normal-mode (or spectral) stability theory of spatially periodic states is generally well understood but a wave packet analysis would provide significant new information about the long-time asymptotics.

In some cases spatially quasi-periodic states can be studied using the Floquet theory and, therefore, the analysis of §§ 2–5 would be applicable. For example, the spatially quasi-periodic states of the real Ginzburg–Landau equation (cf. Bridges & Rowlands 1994) have been analysed using a Floquet decomposition and shown to be unstable but the long-time asymptotics in this case have not been studied.

If the basic state is periodic in only one space dimension the theory of §§ 2–5 can be generalized to the case where the perturbation class depends also on a transverse spatial coordinate (the basic state would be homogeneous in this direction). These are three-dimensional wave packets in a basic flow which is spatially periodic in only one space dimension. Such a generalization in the spatially homogeneous case has been given by Brevdo (1991) and can be carried out along the same lines for the spatially periodic case. A more difficult yet interesting problem would be to consider the absolute and convective instabilities of basic states that are periodic in two space dimensions, as such states and their linear stability are of fundamental importance for pattern formation. A direct generalization of the developed theory to such states will require different methods because in general the Floquet *transformation* does not carry over – although the Floquet *decomposition* (cf. Kuchment 1993) carries over under suitable hypotheses – to the case of PDEs with coefficients periodic in two spatial directions.

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## References

Aranson, I. S., Aranson, L., Kramer, L. & Weber, A. 1992 Stability limits of spirals and travelling waves in non-equilibrium media. *Phys. Rev.* A **46**(6), R2992–R2995.

Benjamin, T. B. 1961 The development of three-dimensional disturbances in an unstable film of liquid flowing down an inclined plane. *J. Fluid Mech.* **10**, 401–419.

Bers, A. 1983 Space-time evolution of plasma instabilities – absolute and convective. In *Handbook of plasma physics* (ed. M. N. Rosenbluth & R. Z. Sagdeev), vol. 1, pp. 451–517. Amsterdam: North-Holland.

Brevdo, L. 1988 A study of absolute and convective instabilities with an application to the Eady model. *Geophys. Astrophys. Fluid Dyn.* **40**, 1–92.

Brevdo, L. 1991 Three-dimensional absolute and convective instabilities, and spatially amplifying waves in parallel shear flows. *ZAMP* **42**, 911–942.

Brevdo, L. 1992 Spatially amplifying waves in plane Poiseuille flow. *ZAMM* **72**(3), 163–174.

Brevdo, L., Bohnsack, E. & Oertel Jr, H. 1994 Absolute und konvektive Orr–Sommerfeld–Stabilitätsanalyse von Nachlaufprofilen eines Kraftfahrzeugs. *ZAMM* **74**(7), 243–256.

Bridges, T. J. & Rowlands, G. 1994 Instability of spatially quasi-periodic states of the Ginzburg–Landau equation. *Proc. R. Soc. Lond.* A **444**, 347–362.

Briggs, R. J. 1964 *Electron-stream interaction with plasmas.* MIT Press.

DiPrima, R. C., Eckhaus, W. & Segel, L. A. 1971 Nonlinear wavenumber interaction in near-critical two-dimensional flows. *J. Fluid Mech.* **49**, 705–744.

Drazin, P. G. & Reid, W. H. 1981 *Hydrodynamic stability.* Cambridge University Press.

Eckhaus, W. 1965 *Studies in nonlinear stability theory.* New York: Springer.

Eckhaus, W. 1993 The Ginzburg–Landau manifold is an attractor. *J. Nonlinear Sci.* **3**, 329–348.

Gaster, M. 1968 The development of three-dimensional wave packets in a boundary layer. *J. Fluid Mech.* **32**(1), 173–184.

Gaster, M. 1975 A theoretical model of a wave packet in the boundary layer on a flat plate. *Proc. R. Soc. Lond.* A **347**, 271–289.

Gaster, M. & Grant, I. 1975 An experimental investigation of the formation and development of a wave packet in a laminar boundary layer. *Proc. R. Soc. Lond.* A **347**, 253–269.

Huerre, P. & Monkewitz, P. A. 1985 Absolute and convective instabilities in free shear layers. *J. Fluid Mech.* **159**, 151–168.

Huerre, P. & Monkewitz, P. A. 1990 Local and global instabilities in spatially developing flows. *A. Rev. Fluid Mech.* **22**, 473–537.

Hunt, R. E. & Crighton, D. G. 1991 Instability of flows in spatially developing media. *Proc. R. Soc. Lond.* A **435**, 109–128.

Infeld, E. & Rowlands, G. 1990 *Nonlinear waves, solitons and chaos.* Cambridge University Press.

Koch, W. 1985 Local instability characteristics and frequency determination of self-excited wake flows. *J. Sound Vib.* **99**, 53–83.

Körner, T. W. 1988 *Fourier analysis.* Cambridge University Press.

Koschmieder, E. L. 1974 Benárd convection. In *Advances in chemical physics* (ed. I. Prigogine & S. A. Rice), vol. 26, pp. 177–212. New York: Wiley-Intersciences.

Kuchment, P. 1993 *Floquet theory for partial differential equations.* Basel: Birkhäuser Verlag.

Landau, L. D. & Lifshitz, E. M. 1953 *Electrodynamics of continuous media.* Moscow: GITTL.

Maginu, K. 1978 Stability of periodic travelling wave solutions of a nerve conduction equation. *J. Math. Biol.* **6**, 49–57.

Polovin, R. V. & Demutskii, V. P. 1981 Character of decay instability. *Sov. Phys. JETP* **54**(5), 906–909.

Stewartson, K. & Stuart, J. T. 1971 A non-linear instability theory for a wave system in plane Poiseuille flow. *J. Fluid. Mech.* **48**, 529–545.

Stuart, J. T. & DiPrima, R. C. 1978 The Eckhaus and Benjamin–Feir resonance mechanisms. *Proc. R. Soc. Lond.* A **362**, 27–41.

Swinton, J. & Elgin, J. 1990 Stability of travelling pulse solutions to a laser equation. *Phys. Lett.* A **145**, 428–433.

Twiss, R. Q. 1951 On oscillations in electron streams. *Proc. Phys. Soc. Lond.* B **64**, 654–665.

Yakubovich, V. A. & Starzhinskii, V. M. 1975 *Linear differential equations with periodic coefficients.* New York: Wiley.

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